Physics Honours & Graduate Course, 2013

University of Western Australia, Australian National University, Monash University

Relativistic Electrodynamics -- Assignment 1

The assignments are of equal value.

Deadline for UWA students: Thursday 18 April, 5 pm, in labelled box on the 1st Floor.

Attempt all questions. The three questions are worth 10 marks each.

Any combination of computer and traditional mathematical methods may be used.

A1. Phase space equations. Let $\nabla_{\mathbf{r}}$, $\nabla_{\mathbf{v}}$, and ∇_{ph} denote the del (gradient) operators in coordinate space, velocity space, and 6-dimensional phase space:

$$\mathbf{r} = (x, y, z)\ , \quad \mathbf{v} \equiv \mathrm{d}\mathbf{r}/\mathrm{dt} = (v_x, v_y, v_z)\ , \quad (\mathbf{r}, \mathbf{v}) = (x, y, z, v_x, v_y, v_z)\ .$$

Note that, in phase space, $\mathbf{r}$ and $\mathbf{v}$ are treated as independent variables. Let $f(\mathbf{r}, \mathbf{v}, t)$ denote a distribution function expressing the probability density of a set of particles in phase space. The time evolution of f is described by the Vlasov equation:

$$\mathrm{D}f/\mathrm{d}t = (\partial/\partial t + \mathbf{v}\bullet\nabla_{\mathbf{r}} + \mathbf{a}\bullet\nabla_{\mathbf{v}})f = 0\ , \qquad \text{with } \mathbf{a} \equiv \mathrm{d}\mathbf{v}/\mathrm{d}t\ .$$

(a) Use a Taylor expansion to define the derivative $\mathrm{D}f/\mathrm{d}t$ as shown above. Comment on why $\mathrm{D}f/\mathrm{d}t$ is the rate of change of f along a particle path in phase space.

> **Note:** The constancy of f is Liouville's theorem: In the absence of sources or sinks, the probability density for finding a particle in an element $\mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{v}$ of phase space volume remains constant along the particle's trajectory through phase space.

(b) Suppose that the particles are all of mass m and charge e moving in an electromagnetic field $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$. Show that:

$$(\mathbf{v}\bullet\nabla_{\mathbf{r}})f = \nabla_{\mathbf{r}}\bullet(\mathbf{v}f) \quad \text{and} \quad (\mathbf{E}\bullet\nabla_{\mathbf{v}})f = \nabla_{\mathbf{v}}\bullet(\mathbf{E}f)\ .$$

(c) Show that, in this electromagnetic case, the Vlasov equation can be written in the form of a continuity equation (conservation law) in phase space:

$$\partial f/\partial t + \nabla_{\mathbf{r}}\bullet(\mathbf{v}f) + \nabla_{\mathbf{v}}\bullet(\mathbf{a}f) = 0 \quad \text{or} \quad \partial f/\partial t + \nabla_{\mathrm{ph}}\bullet[(\mathbf{v}, \mathbf{a})f] = 0$$

with $(\mathbf{v}, \mathbf{a}) = (v_x, v_y, v_z, a_x, a_y, a_z)$ a vector in 6-dimensional phase space.

See over for Problems A2 & A3 . . .

A2. **Some field geometry.** In a particular inertial frame, uniform electric and magnetic fields **E** and **B** make an angle α with each other. It is convenient to introduce the parameter $\varepsilon \equiv E/(cB)$ as a dimensionless measure of the relative strengths of **E** & **B**. Find a Lorentz transformation to another frame in which **E** & **B** are parallel. Let $c\beta$ denote the speed of the second frame relative to the first and solve for β as a function of α and ε. Are there any circumstances in which **E** and **B** can not be made parallel?

A3. **Fields of wires.** The electric field of a long straight wire, carrying charge λ per unit length, is radial with strength at radial distance r from the centre of the wire:

$$E = 2K\lambda/r\,, \qquad K \equiv 1/(4\pi\varepsilon_0) \text{ in SI units.}$$

Make a Lorentz transformation to a frame moving at speed v parallel to the wire. Calculate the electric and magnetic fields in the new frame and compare the magnetic field with that of a long straight wire carrying a current I:

$$\mathbf{B} = (\mu_0/4\pi)2I\,\mathbf{k}/r\,, \qquad \mathbf{k} = \text{unit toroidal vector.}$$

Comment on the physical difference between a current-carrying wire and a charged wire moving in the direction of its length.

RB, 08 April 2013