Relativistic Electrodynamics 2013

Solutions to Exercises 1 and 3–6 in the RED notes

Exercise 1 (p RED6). Charge invariance test. Imagine that charge were to vary with the speed of the charge carrier in the same way as does mass; *i.e.* according to the Lorentz factor $\gamma \equiv (1 - v^2/c^2)^{-1/2}$.

(A) Estimate, to order of magnitude, the difference in net charge that might be expected between a hydrogen molecule and a helium atom. We can assume that the proton's kinetic energy increases from something negligible to a few MeV in going from the H_2 molecule to the He nucleus, and that the KE of the two electrons increases by some tens of eV in going from H_2 to He. FYI:

- The dissociation (or binding) energy of H_2 – the energy to separate it into two H atoms – is 4.5 eV; the inter-nuclear separation is 0.74 Å and the first ionization potential is 15.4 eV.
- The binding energy of the He nucleus is 7 MeV per nucleon; so, from the virial theorem for a bound system, we can take the KE of a nucleon to be a few MeV. The ionization potentials for helium are 24.6 & 54.4 eV.

(B) If it happened that H_2 were exactly neutral, how much excess charge would you expect to find in 1 litre of helium gas at STP? (1 mole has a volume of 22.4 litre at STP and contains 6.023×10^{23} molecules.) What would be the electric field strength just outside the (non-metallic) container?

Solution: (A) The particles all have $v^2/c^2 << 1$ so KE $\approx \frac{1}{2} m_0 v^2$ and $\gamma \approx 1 + \frac{1}{2} v^2/c^2$, giving

$$\mathrm{KE}/(m_0 c^2) \approx \tfrac{1}{2} v^2/c^2 \approx \gamma - 1 .$$

For a proton going from the H_2 molecule (negligible KE) to the He nucleus (KE a few MeV):

$$\Delta(\mathrm{KE})/(m_0 c^2) \approx (\text{a few MeV})/(1\ \text{GeV}) \approx (\text{say})\ 3 \times 10^{-3},$$

so proton in He has: $\gamma - 1 \approx 3 \times 10^{-3}$.

There are 2 protons so, if $q \propto \gamma$, the charge variation (H_2 to He) due to the protons would be

$$\Delta q \approx +6 \times 10^{-3} e, \qquad e = \text{proton (proper or rest) charge.}$$

For each of the two electrons, Δ(KE) $\approx$ (say) 30 eV, so:

$$\Delta(\mathrm{KE})/(m_0 c^2) \approx (30\ \text{eV})/(0.5\ \text{MeV}) \approx 6 \times 10^{-5},$$

giving $\Delta(\gamma - 1) \approx 6 \times 10^{-5}$ and so (2 electrons) $\Delta q \approx -1 \times 10^{-4} e$. Hence we might expect a net charge variation per atom or molecule of:

$$\Delta q \sim 10^{-2} e, \text{ essentially due to the protons.}$$

(B) 1 mole has a volume of 22.4 litre at STP and contains 6.023×10^{23} molecules. So:

1 L contains (1/22.4 mole)(6.023×10^{23} molecules/mole) = 3×10^{22} molecules or atoms.

So for 1 L: $\Delta q \sim (3 \times 10^{22})(+6 \times 10^{-3})(1.6 \times 10^{-19}\ \text{C}) \sim 30\ \text{C}$.

Take the litre to be roughly a sphere of radius $\approx$ 6 cm. The electric field strength at the surface is:

$$E \sim K \Delta q / r^2 \sim (9 \times 10^9\ \text{N m}^2\ \text{C}^{-1})(30\ \text{C})/(6 \times 10^{-2}\ \text{m}) \sim 10^{14}\ \text{V/m} \,!$$

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<u>Exercise 3 (p RED20)</u>. Lorenz & Coulomb gauge conditions. By direct Lorentz transformation of the potentials and partial derivatives check that the Lorenz condition $\nabla\bullet\mathbf{A} + c^{-2}\partial\phi/\partial t = 0$ is Lorentz invariant but that the Coulomb gauge condition $\nabla\bullet\mathbf{A} = 0$ is not.

Solution: The components of the 4-potential $(\Phi^\mu) = (\phi, c\mathbf{A})$ transform in the same way as the x^μ (*i.e.* ct, x, y, z -- the components of the prototype <u>contravariant</u> 4-vector):

$$ct',\ x'_1,\ x'_2,\ x'_3 \ = \ \gamma(ct - \beta x_1),\ \gamma(x_1 - \beta ct),\ x_2,\ x_3 \qquad \beta \equiv v/c,\ \gamma \equiv (1-\beta^2)^{-1/2}\,.$$

so $$\phi',\ A'_1,\ A'_2,\ A'_3 \ = \ \gamma(\phi - vA_1),\ \gamma(A_1 - c^{-1}\beta\phi),\ A_2,\ A_3\,.$$

The 4-dimensional gradient operator is the prototype <u>covariant</u> vector -- its component transform according to the <u>inverse</u> Lorentz Transformation:

$$\partial_0',\ \partial_1',\ \partial_2',\ \partial_3' \ = \ \gamma(\partial_0 + \beta\partial_1),\ \gamma(\partial_1 + \beta\partial_0),\ \partial_2,\ \partial_3\,.$$

From these:

$$\begin{aligned}\nabla'\bullet\mathbf{A}' &= \gamma^2(\partial_1 + \beta\partial_0)(A_1 - c^{-1}\beta\phi) + \partial_2A_2 + \partial_3A_3 \\ &= \gamma^2\big[\ (\partial_1 + \beta\partial_0)A_1 - c^{-1}(\beta\partial_1 + \beta^2\partial_0)\phi\ \big] + \partial_2A_2 + \partial_3A_3 \qquad (1)\end{aligned}$$

and $$\begin{aligned}c^{-2}\partial V'/\partial t' &\equiv c^{-1}\partial_0'\,\phi' = c^{-1}\gamma^2\partial(\partial_0 + \beta\partial_1)(\phi - vA_1) \\ &= \gamma^2\big[\ c^{-1}(\partial_0 + \beta\partial_1)\phi - (\beta\partial_0 + \beta^2\partial_1)A_1\ \big]\,. \qquad (2)\end{aligned}$$

Adding Eqs (1) & (2) gives:

$$\begin{aligned}\nabla'\bullet\mathbf{A}' + c^{-2}\partial V'/\partial t' &= \gamma^2\big[\ (1-\ \beta^2)\partial_1A_1 + c^{-1}(1-\ \beta^2)\partial_0\phi\ \big] + \partial_2A_2 + \partial_3A_3 \\ &= \gamma^2\big[\ (1-\ \beta^2)\partial_1A_1 + c^{-1}(1-\ \beta^2)\partial_0\phi\ + \partial_2A_2 + \partial_3A_3\ \big]\,. \\ &= \partial_1A_1 + c^{-1}\partial_0\phi\ + \partial_2A_2 + \partial_3A_3 \qquad \text{(using } \gamma^{-2} \equiv 1-\beta^2\text{)} \\ &= \nabla\bullet\mathbf{A} + c^{-2}\partial\phi/\partial t\,,\ \text{as required.}\end{aligned}$$

From Eq (1): $$\begin{aligned}\nabla'\bullet\mathbf{A}' &= \ \gamma^2\partial_1A_1 + \partial_2A_2 + \partial_3A_2 + \gamma^2\big[\ \beta\partial_0A_1 - c^{-1}(\beta\partial_1 + \beta^2\partial_0)\phi\ \big] \\ &= \nabla\bullet\mathbf{A} + (\gamma^2 - 1)\partial_1A_1 + \gamma^2\beta\big[\ \partial_0A_1 - c^{-1}(\partial_1 + \beta\partial_0)\phi\ \big] \\ &\neq \nabla\bullet\mathbf{A}\ \ \text{in general; } \textit{i.e.} \text{ the choice } \nabla\bullet\mathbf{A} = 0 \text{ is not Lorentz invariant.}\end{aligned}$$

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Exercise 4A, 4B (p RED26). Field transformation. Verify the transformation rules for **E** & **B**, in both component and vector forms, as given in the RED notes (p RED25).

Solution: Begin by recalling that, for transformations between Lorentz frames in standard configuration, the contravariant components of a 4-vector (V^μ) follow the rule:

$$V^{0\prime},\ V^{1\prime},\ V^{2\prime},\ V^{3\prime} = \gamma(V^0 - \beta V^1),\ (V^1 - \beta V^0),\ V^2,\ V^3$$

[*e.g.* (ct, **r**), the prototype contravariant 4-vector]. Therefore, as $(\Phi^\mu) \equiv (\phi, c\mathbf{A})$ is a contravariant 4-vector [note that $A^k = A_k$ -- no distinction between contravariance and covariance for the 3-vector]:

$$\phi',\ A_1',\ A_2',\ A_3' = \gamma(\phi - vA_1),\ \gamma(A_1 - \beta\phi/c),\ A_2,\ A_3$$

-- establishing the transformation rules for the electromagnetic scalar and vector potentials.

Magnetic field: Use $\mathbf{B}' = \nabla' \times \mathbf{A}'$ in component form:

$$B'_1 = \frac{\partial A'_3}{\partial x'^2} - \frac{\partial A'_2}{\partial x'^3} = \frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3} = B_1 \text{ -(i)} \quad \left(\begin{array}{l} x'^2 = x^2,\ x'^3 = x^3 \\ A'_2 = A_2,\ A'_3 = A_3 \end{array} \right.$$

$$B'_2 = \frac{\partial A'_1}{\partial x'^3} - \frac{\partial A'_3}{\partial x'^1} = \gamma\frac{\partial}{\partial x^3}(A_1 - \beta\phi/c) - \frac{\partial A_3}{\partial x'^1}. \quad \left(\begin{array}{l} x'^3 = x^3 \\ A'_1 = \gamma(A_1 - \beta\phi/c) \\ A'_3 = A_3,\ \gamma \text{ const} \end{array} \right.$$

From calculus, changing the independent variables in a function of several variables:

$$\frac{\partial}{\partial x'^1} = \frac{\partial x^\mu}{\partial x'^1}\frac{\partial}{\partial x^\mu} = \frac{\partial x^0}{\partial x'^1}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^1}\frac{\partial}{\partial x^1} + \underbrace{0 + 0}_{x^2 \,\&\, x^3 \text{ indept of } x'^1}$$

$$= \gamma(\beta\, \partial/\partial x^0 + \partial/\partial x^1). \text{ ——— (1)}$$

So

$$B'_2 = \gamma(A_{1,3} - A_{3,1}) - \gamma\beta(\phi_{,3} + \partial A_3/\partial t)/c = \gamma(B_2 + vE_3/c^2) \text{ —(ii)}$$

$$B'_3 = \frac{\partial A'_2}{\partial x'^1} - \frac{\partial A'_1}{\partial x'^2} = \frac{\partial A_2}{\partial x'^1} - \gamma\frac{\partial}{\partial x^2}(A_1 - \beta\phi/c) \quad \left(\begin{array}{l} A'_2 = A_2 \\ A'_1 = \gamma(A_1 - \beta\phi/c) \\ x'^2 = x^2,\ \gamma \text{ const} \end{array} \right.$$

$$\overset{(1)}{=} \gamma(A_{2,1} - A_{1,2}) + (\gamma\beta/c)(\partial\phi/\partial x^2 + \partial A_2/\partial t)$$

$$= \gamma(B_3 - vE_2/c^2). \text{ ——— (iii)}$$

Since $-\underset{\sim}{v} \times \underset{\sim}{E} = (0, vE_3, -vE_2)$, eqs (i)-(iii) can be expressed as

$$\underset{\sim}{B}'_\parallel = \underset{\sim}{B}_\parallel \quad \& \quad \underset{\sim}{B}'_\perp = \gamma(\underset{\sim}{B} - \underset{\sim}{v} \times \underset{\sim}{E}/c^2)_\perp. \quad \left(\begin{array}{l} \parallel \,\&\, \perp \text{ rel.} \\ \text{direction of} \\ \text{motion of frames} \end{array} \right.$$

Electric field. Use $\underset{\sim}{E}' = -\nabla'\phi' - \partial\underset{\sim}{A}'/\partial t'$ in c/ct form.

First note that [using (1), $x'^2 = x^2$, $x'^3 = x^3$ & $\phi' = \gamma(\phi - v A_1)$]

$$\nabla'\phi' = \gamma\left(\gamma\left[\frac{\beta}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x^1}\right], \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)\left(\phi - v A_1\right). \quad (\alpha)$$

As with the derivation of (1):

$$\frac{\partial}{\partial t'} = \frac{\partial x^\mu}{\partial t'}\frac{\partial}{\partial x^\mu} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t} + \frac{\partial x^1}{\partial t'}\frac{\partial}{\partial x^1} + 0 + 0$$

$$= \gamma\left(\partial/\partial t + v\,\partial/\partial x^1\right). \quad \text{—— (2)}$$

Using (2) & the transformation of $\underset{\sim}{A}$:

$$\partial\underset{\sim}{A}'/\partial t' = \gamma\left(\partial/\partial t + v\,\partial/\partial x^1\right)\left(\gamma\left[A_1 - \beta\phi/c\right], A_2, A_3\right). \quad (\beta)$$

From (α) & (β):

$$E'_1 = -\gamma^2\left[\left(\cancel{\frac{\beta}{c}\frac{\partial}{\partial t}} + \frac{\partial}{\partial x^1} - \cancel{\frac{\beta}{c}\frac{\partial}{\partial t}} - \beta^2\frac{\partial}{\partial x^1}\right)\phi + \left(-\beta^2\frac{\partial}{\partial t} - \cancel{v\frac{\partial}{\partial x^1}} + \frac{\partial}{\partial t} + \cancel{v\frac{\partial}{\partial x^1}}\right)A_1\right]$$

$$= -\gamma^2\left(1-\beta^2\right)\left(\partial\phi/\partial x^1 + \partial A_1/\partial t\right) = E_1 \quad \text{—(iv)} \quad \left(1-\beta^2 = \gamma^{-2}\right.$$

$$E'_2 = -\gamma\left[\frac{\partial\phi}{\partial x^2} + \frac{\partial A_2}{\partial t} + v\left(A_{2,1} - A_{1,2}\right)\right] = \gamma\left(E_2 - v B_3\right) \quad \text{—(v)}$$

$$E'_3 = -\gamma\left[\frac{\partial\phi}{\partial x^3} + \frac{\partial A_3}{\partial t} - v\left(A_{1,3} - A_{3,1}\right)\right] = \gamma\left(E_3 + v B_2\right). \quad \text{—(vi)}$$

Since $\underset{\sim}{v} \times \underset{\sim}{B} = \left(0, -v B_3, v B_2\right)$, eqs (iv)–(vi) can be expressed as

$$\underset{\sim}{E}'_{\parallel} = \underset{\sim}{E}_{\parallel} \quad \& \quad \underset{\sim}{E}'_{\perp} = \gamma\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right)_{\perp}$$

Exercise 5 (p RED27). Field Invariants of the electromagnetic field. Verify by direct calculation that $\mathbf{E}\bullet\mathbf{B}$ and $E^2 - c^2B^2$ are Lorentz invariants.

Solution: The transformation rules for the field components are:

$$E_{\parallel}' = E_{\parallel}, \qquad \mathbf{E}_{\perp}' = \gamma(v)(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\perp}, \tag{1a,b}$$

$$B_{\parallel}' = B_{\parallel}, \qquad \mathbf{B}_{\perp}' = \gamma(v)(\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2)_{\perp}, \tag{2a,b}$$

for components parallel & perpendicular to $\mathbf{v}$, the velocity of S′ relative to S.

Invariance of $\mathbf{E}\bullet\mathbf{B}$

$$\underset{\sim}{E}'\cdot\underset{\sim}{B}' \equiv E_{\parallel}'B_{\parallel}' + \underset{\sim}{E}_{\perp}'\cdot\underset{\sim}{B}_{\perp}'$$

$$\overset{(1,2)}{=} E_{\parallel}B_{\parallel} + \gamma^2(v)\left[\underset{\sim}{E}_{\perp} + (\underset{\sim}{v}\times\underset{\sim}{B})_{\perp}\right]\cdot\left[\underset{\sim}{B}_{\perp} - (v\times\underset{\sim}{E})_{\perp}/c^2\right]$$

$$= E_{\parallel}B_{\parallel} + \gamma^2(v)\left[\underset{\sim}{E}_{\perp} + \underset{\sim}{v}\times\underset{\sim}{B}_{\perp}\right]\cdot\left[\underset{\sim}{B}_{\perp} - (\underset{\sim}{v}\times\underset{\sim}{E}_{\perp})/c^2\right]$$

Since $(\underset{\sim}{v}\times\underset{\sim}{B})_{\perp} = \underset{\sim}{v}\times\underset{\sim}{B} = \underset{\sim}{v}\times\underset{\sim}{B}_{\perp}$ + sim. for $\underset{\sim}{E}$.

Thus:

$$\underset{\sim}{E}'\cdot\underset{\sim}{B}' = E_{\parallel}B_{\parallel} + \gamma^2\underset{\sim}{E}_{\perp}\cdot\underset{\sim}{B}_{\perp} - \gamma^2\underbrace{(\underset{\sim}{v}\times\underset{\sim}{B}_{\perp})\cdot(\underset{\sim}{v}\times\underset{\sim}{E}_{\perp})}_{(3)}/c^2$$

But, using cartesian components

$$(\underset{\sim}{v}\times\underset{\sim}{B}_{\perp})\cdot(\underset{\sim}{v}\times\underset{\sim}{E}_{\perp}) = (0, -vB_z, vB_y)\cdot(0, -vE_z, vE_y)$$

$$= v^2(B_zE_z + B_yE_y) = v^2\underset{\sim}{E}_{\perp}\cdot\underset{\sim}{B}_{\perp}$$

So Eq(3) becomes

$$\underset{\sim}{E}'\cdot\underset{\sim}{B}' = E_{\parallel}B_{\parallel} + \gamma^2\underset{\sim}{E}_{\perp}\cdot\underset{\sim}{B}_{\perp}(1 - v^2/c^2)$$

$$= E_{\parallel}B_{\parallel} + \underset{\sim}{E}_{\perp}\cdot\underset{\sim}{B}_{\perp} \qquad (1 - v^2/c^2 \equiv \gamma^{-2}$$

$$= \underset{\sim}{E}\cdot\underset{\sim}{B} \qquad \text{ie invariant}$$

Invariance of $E^2 - c^2B^2$

$$E_\perp'^{\,2} \overset{(1b)}{=} \gamma^2(v)\left(\underset{\sim}{E}_\perp + \underset{\sim}{v}\times\underset{\sim}{B}_\perp\right)^2 \qquad \left((\underset{\sim}{v}\times\underset{\sim}{B})_\perp = \underset{\sim}{v}\times\underset{\sim}{B} = \underset{\sim}{v}\times\underset{\sim}{B}_\perp \right.$$

& $$B_\perp'^{\,2} \overset{(2b)}{=} \gamma^2(v)\left(\underset{\sim}{B}_\perp - \underset{\sim}{v}\times\underset{\sim}{E}_\perp/c^2\right)^2 \qquad \text{\& sim for } \underset{\sim}{E}$$

So
$$E_\perp'^{\,2} = \gamma^2\Big[E_\perp^2 + \overbrace{2\,\underset{\sim}{v}\times\underset{\sim}{B}_\perp\cdot\underset{\sim}{E}_\perp}^{=-2\underset{\sim}{v}\times\underset{\sim}{E}_\perp\cdot\underset{\sim}{B}_\perp} + (\underset{\sim}{v}\times\underset{\sim}{B}_\perp)^2\Big]$$

& $$c^2B_\perp'^{\,2} = \gamma^2\Big[c^2B_\perp^2 - 2\,\underset{\sim}{v}\times\underset{\sim}{E}_\perp\cdot\underset{\sim}{B}_\perp + (\underset{\sim}{v}\times\underset{\sim}{E}_\perp)^2/c^2\Big]$$

But $(\underset{\sim}{v}\times\underset{\sim}{B}_\perp)^2 = v^2B_\perp^2$ & sim for $\underset{\sim}{E}_\perp$.

Hence
$$E_\perp'^{\,2} = \gamma^2\Big[E_\perp^2 + v^2B_\perp^2 - 2\,\underset{\sim}{v}\times\underset{\sim}{E}_\perp\cdot\underset{\sim}{B}_\perp\Big]$$

& $$c^2B_\perp'^{\,2} = \gamma^2\Big[c^2B_\perp^2 + v^2E_\perp^2/c^2 - 2\,\underset{\sim}{v}\times\underset{\sim}{E}_\perp\cdot\underset{\sim}{B}_\perp\Big]$$

So
$$E_\perp'^{\,2} - c^2B_\perp'^{\,2} = \gamma^2\left(E_\perp^2 - c^2B_\perp^2\right)\left(1 - v^2/c^2\right)$$
$$= E_\perp^2 - c^2B_\perp^2 \qquad \left(1 - v^2/c^2 \equiv \gamma^{-2}(v)\right.$$

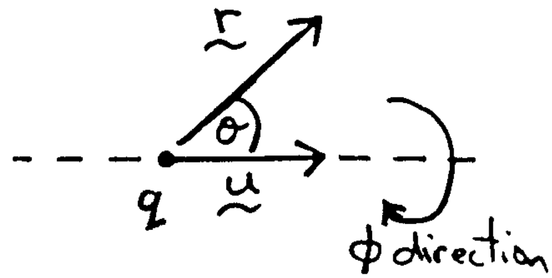
Also $$E_\parallel'^{\,2} - c^2B_\parallel'^{\,2} \overset{(1a),(2a)}{=} E_\parallel^2 - c^2B_\parallel^2$$

So
$$E'^2 - c^2B'^2 = E^2 - c^2B^2$$
ie invariant

Exercise 6 (p RED34). Uniformly moving charge. Evaluate $\nabla\bullet\mathbf{E}$ & $\nabla\times\mathbf{E}$ for a uniformly moving charge in cylindrical and spherical polar coordinates, with axes along the velocity. As $\nabla\times\mathbf{E}\neq\mathbf{0}$, the field is not electrostatic – as expected since there is a $\mathbf{B}$ field produced as well.

Solution: In the notation of the diagram, with $\mathbf{r}$ radially away from the charge and $\hat{\mathbf{r}}\equiv\mathbf{r}/r$, we found in lectures that, with $K\equiv 1/(4\pi\varepsilon_0)$ in SI units:

$$\mathbf{E} = E\,\hat{\mathbf{r}} \text{ with } E = (Kq/\gamma^2r^2)(1-\beta^2\sin^2\theta)^{-3/2}. \qquad (1)$$



We first use spherical polar coordinates (r, θ, ϕ) about the line of motion of the charge as axis. The div and curl of a vector $\mathbf{a} = a_r\hat{\mathbf{r}}$, with a spherical polar radial component only, are:

$$\nabla\bullet\mathbf{a} = r^{-2}\partial_r(r^2a_r) \qquad \text{and} \qquad \nabla\times\mathbf{a} = \big(0,\ (r\sin\theta)^{-1}\partial_\phi a_r,\ -r^{-1}\partial_\theta a_r\big)\,.$$

As $\mathbf{E} = E\,\hat{\mathbf{r}}$ and the radial dependence of E is $E\propto r^{-2}$, we see that $\nabla\bullet\mathbf{E} = 0$. As

$$\partial_\phi E = 0 \qquad \text{and} \qquad \partial_\theta E = (3\beta^2/2)(\sin 2\theta)(Kq/\gamma^2r^2)(1-\beta^2\sin^2\theta)^{-5/2}$$

we see that $\nabla\times\mathbf{E}$ forms loops around the direction of motion:

$$\nabla\times\mathbf{E} = -(Kq/\gamma^2r^3)(3\beta^2/2)[(\sin 2\theta)/(1-\beta^2\sin^2\theta)^{5/2}]\,\hat{\boldsymbol{\phi}}\,. \qquad (2)$$

We now use cylindrical polar coordinates (ϖ, ϕ, z) about the line of motion of the charge as z axis. The div and curl of a vector with no azimuthal component, $\mathbf{a} = (a_\varpi, 0, a_z)$, are:

$$\nabla\bullet\mathbf{a} = \varpi^{-1}\partial_\varpi(\varpi a_\varpi) \ + \partial_z a_z \qquad \text{and} \quad \nabla\times\mathbf{a} = \big(\varpi^{-1}\partial_\phi a_z,\ \partial_z a_\varpi - \partial_\varpi a_z,\ -\varpi^{-1}\partial_\phi a_\varpi\big)\,.$$

We now partially transform the function in the denominator of E, Eq (1), from (r, θ) to (ϖ, z) via

$$\varpi = r\sin\theta, \qquad z = r\cos\theta \qquad (3)$$

to

$$r^2(1-\beta^2\sin^2\theta)^{3/2} = r^{-1}(r^2-\beta^2r^2\sin^2\theta)^{3/2} = r^{-1}[(1-\beta^2)\varpi^2+z^2]^{3/2}\,. \qquad (4)$$

Hence we get E/r in terms of the cylindrical coordinates:

$$E/r = (Kq/\gamma^2)[(1-\beta^2)\varpi^2+z^2]^{-3/2}\,.$$

In these coordinates, Eq (1) takes the form:

$$\mathbf{E} = (E\sin\theta, 0, E\cos\theta) = (E\varpi/r, 0, Ez/r) = (Kq/\gamma^2)[(1-\beta^2)\varpi^2+z^2]^{-3/2}(\varpi, 0, z)\,.$$

Hence the derivatives:

$$(Kq/\gamma^2)^{-1}\partial_\varpi E_\varpi \ = \ [(1-\beta^2)\varpi^2+z^2]^{-3/2} \ - \ (3\rho^2)(1-\beta^2)[(1-\beta^2)\rho^2+z^2]^{-5/2}$$

$$(Kq/\gamma^2)^{-1}\partial_z E_z \ = \ [(1-\beta^2)\varpi^2+z^2]^{-3/2} \ - \ (3z^2)[(1-\beta^2)\rho^2+z^2]^{-5/2}$$

$$(Kq/\gamma^2)^{-1}\partial_z E_\varpi \ = \ -(3\varpi z)[(1-\beta^2)\varpi^2+z^2]^{-5/2}$$

$$(Kq/\gamma^2)^{-1}\partial_\varpi E_z \ = \ -(3\varpi z)(1-\beta^2)[(1-\beta^2)\varpi^2+z^2]^{-5/2}$$

plus $\partial_\phi E_\varpi = 0$ and $\partial_\phi E_z = 0$. It quickly follows that $\nabla\bullet\mathbf{E} = 0$ and that

$$\nabla\times\mathbf{E} = -(Kq/\gamma^2)(3\beta^2\varpi z)[(1-\beta^2)\varpi^2+z^2]^{-5/2}\hat{\boldsymbol{\phi}}\,.$$

This readily transforms into the spherical polar form (2) using Eqs (3) and (4).

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