Relativistic Electrodynamics 2013

Solutions to Exercises 8, 9, 10, 11 in the RED notes

Exercise 8 (p RED40). Uniformly moving charge – "flattening" of the E field. The electric field strength of a uniformly moving charge q, of speed $u \equiv c\beta$, is given by (see diagram)

$$E = \frac{Kq}{r^2} \frac{1-\beta^2}{(1-\beta^2 \sin^2\theta)^{3/2}}\,.$$

Let $\theta \equiv (\pi/2) - \varepsilon$. For extremely relativistic motion, $\gamma >> 1$, the flux is concentrated into angles $|\,\varepsilon\,| << 1$. Show that, in this case:

$$E \approx \frac{Kq}{r^2} \frac{\gamma}{(1+\gamma^2\varepsilon^2)^{3/2}} \quad \text{for} \;\; \varepsilon^2 << 1 \;\; \text{and} \;\; \gamma^2 >> 1\,.$$

Solution. In the given equation for the electric field strength, we need to approximate the angular dependence. With $\theta \equiv (\pi/2) - \varepsilon$:

$$\sin^2\theta = \cos^2\varepsilon \approx [1 - (\varepsilon^2/2)]^2 \approx 1 - \varepsilon^2 \quad \text{for} \;\; \varepsilon^2 << 1\,. \tag{1}$$

Noting that $\beta^2 \equiv 1 - \gamma^{-2}$ is often a useful substitution for β at large γ, we see that:

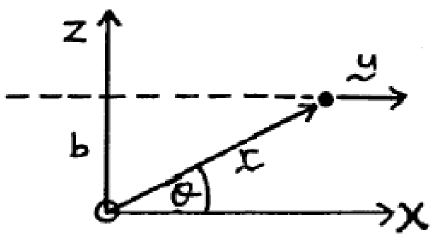
$$1 - \beta^2 \sin^2\theta \approx 1 - (1-\gamma^{-2})(1-\varepsilon^2) \quad \text{for} \;\; \varepsilon^2 << 1 \tag{2}$$

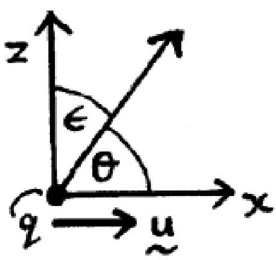
$$\approx \gamma^{-2} + \varepsilon^2 \quad \text{for} \;\; \varepsilon^2 << 1 \;\; \text{and} \;\; \gamma^2 >> 1\,, \tag{3}$$

neglecting the product $\gamma^{-2}\varepsilon^2$ of two small factors. With this result for the angular dependence in the equation for E:

$$E \approx \frac{Kq}{r^2} \frac{\gamma}{(1+\gamma^2\varepsilon^2)^{3/2}} \quad \text{for} \;\; \varepsilon^2 << 1 \;\; \text{and} \;\; \gamma^2 >> 1\,, \tag{4}$$

valid close to the transverse direction ($\varepsilon^2 << 1$) for a highly relativistic ($\gamma^2 >> 1$) charge.





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Exercise 9 (p RED 69). Larmor orbits. Check the formulas for ω_B (the Larmor angular frequency) and r_L (the Larmor radius). Show that the orbiting charge has a magnetic moment of magnitude

$$\mu_L = (1/2)m_e v_\perp^2/B \,.$$

What is the direction of this **Larmor magnetic moment**?

Answer: From standard mechanics, the orbital speed v, orbital angular velocity magnitude ω, and centripetal acceleration magnitude of an orbiting particle are related by

$$v = \omega r \quad \text{and} \quad a = v^2/r = \omega v \,.$$

For an electron (charge $-e$, mass m_e), in a Larmor orbit around a magnetic field of strength B at speed $v_\perp$, these become

$$v_\perp = \omega_B r_L \quad \text{and} \quad a_L = v_\perp^2/r_L = \omega_B v_L \,. \qquad \text{(1a, b)}$$

Equating $\omega_B v_L$ to the magnitude $ev_\perp B/m_e$ of the magnetic force per unit mass on the electron (directed toward the field line) gives

$$\omega_B = eB/m_e \,. \qquad (2)$$

Substituting (1a) into (2) gives the orbit's Larmor radius:

$$r_L = m_e v_\perp/(eB) \,. \qquad (3)$$

The orbiting charge is equivalent to a circulating electric current e/T_L, with T_L the Larmor orbital period:

$$T_L = 2\pi/\omega_B =^{(2)} 2\pi m_e/(eB) \,. \qquad (4)$$

This current loop has magnetic moment of magnitude

$$\mu_L = \text{current} \times \text{area} = [e^2B/(2\pi m_e)](\pi r_L^2) =^{(2)} [e^2B/(2\pi m_e)]\pi[m_e v_\perp/(eB)]^2$$
$$= (1/2)m_e v_\perp^2/B \,.$$

There are

Consider an electron orbiting a $\mathbf{B}$ line with velocity $\mathbf{v}$. Its motion must be in the right-handed sense: $\mathbf{v} \times \mathbf{B}$ is directed outward from $\mathbf{B}$ and the magnetic force is directed toward $\mathbf{B}$ as required. Hence the current is in the left-handed sense and produces a magnetic dipole moment directed against the given magnetic field.

This can also be seen as a result of **Lenz's law**: the motion of an electron in response to the magnetic field must be such as to produce a field contribution that opposes the original field. That is, the reponse is **diamagnetic**.

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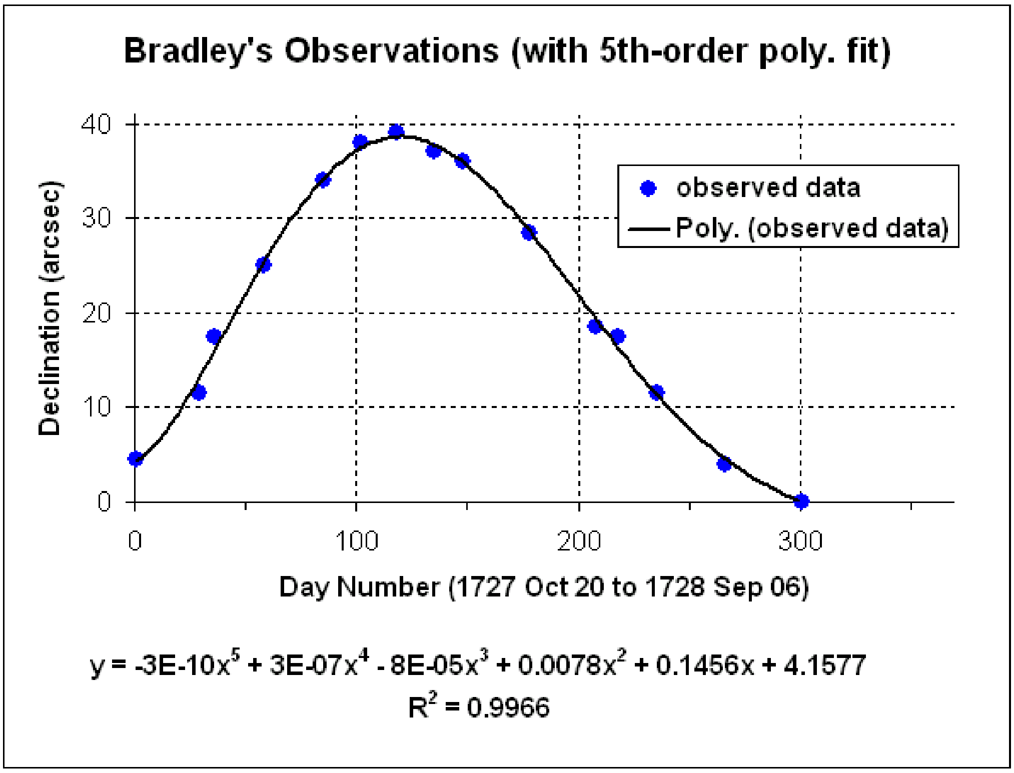
<u>Exercise 10 (p RED 80)</u>. Bradley's measurements. Plot the observed data from Bradley's table and fit to a sine curve.

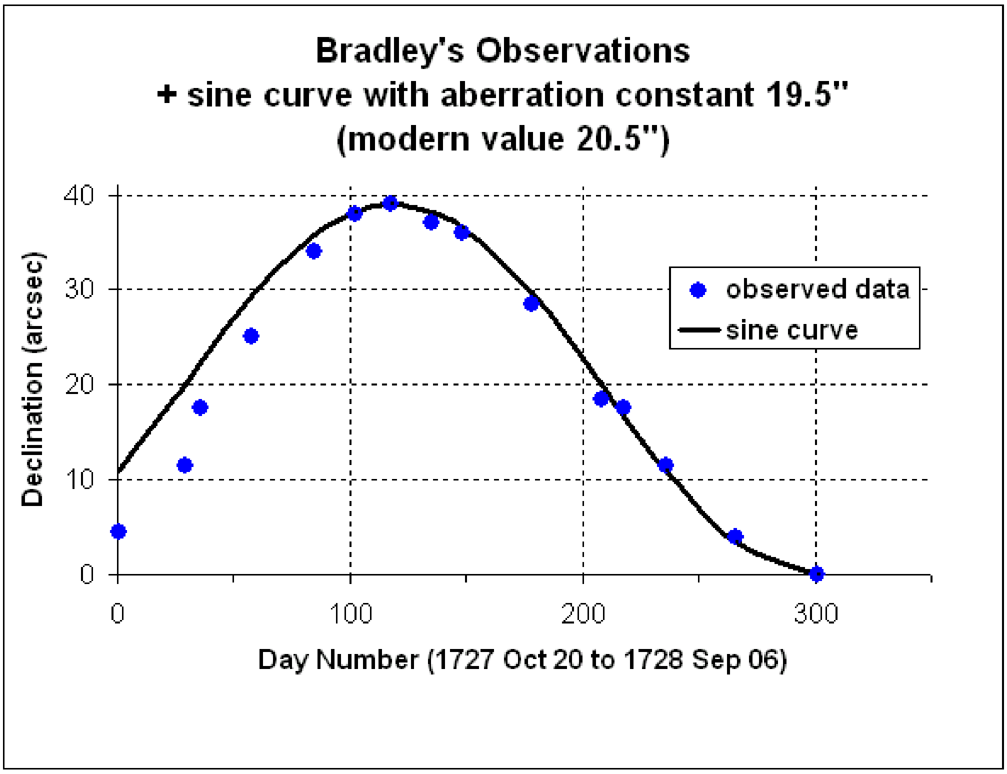
IV. *A Letter from the Reverend Mr.* James Bradley *Savilian Profeſſor of Aſtronomy at* Oxford, *and* F.R.S. *to* Dr.Edmond Halley Aſtronom. Reg. &c. *giving an Account of a new diſcovered Motion of the* Fix'd Stars.

SIR,

YOU having been pleaſed to expreſs your Satisfaction with what I had an Opportunity ſometime ago, of telling you in Converſation, concerning

1727.	D.	The Difference of Declination by Obſervation. "	The Difference of Declination by the Hypotheſis. "	1728.	D.	The Difference of Declination by Obſervation. "	The Difference of Declination by the Hypotheſis. "
October 20th	--	4½	4½	*March*	24	37	38
November	17	11½	12	*April*	6	36	36½
December	6	17½	18½	*May*	6	28½	29½
- - -	28	25	26	*June*	5	18½	20
1728				- -	15	17½	17
January	24	34	34	*July*	3	11½	11½
February	10	38	37	*Auguſt*	2	4	4
March	7	39	39	*September*	6	0	0





The above sine curve is of fixed period (365 days) and amplitude, and is not fitted to the data.

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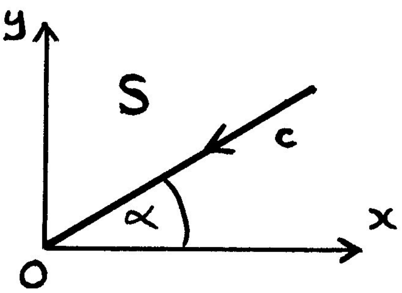
Exercise 11 (p RED 81). The searchlight effect. A light source emits spherically symmetrically in its inertial rest frame S_0, which is moving at speed v toward an observer in inertial frame S, with S & S_0 in standard configuration.

(a) Show that the light in the observer's frame is concentrated into a cone, with half the photons travelling within a cone of half-angle $\alpha = \arccos(v/c)$ toward the observer.

(b) Show that, for highly relativistic motion of the source, $\alpha = \arccos(v/c)$ reduces to $\alpha \approx 1/\gamma$, with γ the Lorentz factor of the source. That is, the light in S is concentrated into a narrow cone, with half the photons travelling in a cone of half-angle $1/\alpha$ toward the observer. This is the **searchlight effect** or **headlight effect**.



Solution: Let the source be at the origin of its inertial rest frame, and regard S as being the moving inertial frame, in standard configuration, as in the diagram. Thus the source is approaching the observer (in S). Apply the aberration formula ($\beta \equiv v/c$):

$$\cos\alpha = (\cos\alpha_0 + \beta)/(1 + \beta\cos\alpha_0). \qquad (1)$$

(a) In S_0, half the photons are emitted (by spherical symmetry) into $-\pi/2 < \alpha_0 < \pi/2$. By (1), the limits of this range correspond in S to

$$\cos\alpha = \beta; \qquad \textit{i.e.}\text{ to} \qquad \alpha = \pm\arccos\beta. \qquad (2)$$

So, for a source directly approaching an observer at speed v, half the photons are observed to be concentrated into a cone of half-angle $\arccos(v/c)$.

This effect, *aberration of light*, is first-order in β and hence can be described as 'semi-relativistic'. It disappears in the strict classical limit, $c \to \infty$, but is the same in non-relativistic theory with finite c. We can see why: the aberration formula (1) comes from the relativistic velocity composition formula:

$$u_x = (u_{x0} + v)/(1 + v_{x0}v/c^2) \qquad \text{with } u_{x0} = c\cos\alpha_0 \ \ \& \ \ u_x = c\cos\alpha$$

representing the x components, in S_0 & S, of the velocity of a light ray (speed c); Eq (1) follows immediately. If, however, we use the non-relativistic velocity addition formula,

$$u_x = u_{x0} + v \qquad \text{with } u_{x0} = c\cos\alpha_0 \ \ \& \ \ u_x = c\cos\alpha,$$

we get

$$\cos\alpha = \cos\alpha_0 + \beta \qquad (3)$$

for the non-relativistic aberration formula, instead of (1).We see that the limits of the range $-\pi/2 < \alpha_0 < \pi/2$ in S_0 correspond in S to $\cos\alpha = \beta$, exactly as in the relativistic treatment.

(b) We now want to look at the $\cos\alpha = \beta$ result, Eq (2), when $\beta \to 1$. Since $\gamma^{-2} \equiv 1 - \beta^2$, we have

$$\beta \equiv (1 - \gamma^{-2})^{1/2} \approx 1 - 1/(2\gamma^2) \qquad \text{for } \gamma^2 >> 1 \qquad \text{(i)}$$

as a useful expression for β for highly relativistic motion. With β close to 1, α will be a small angle, so

$$\cos\alpha \approx 1 - (\alpha^2/2)\,. \qquad \text{(ii)}$$

From Eqs (i) & (ii), $\cos\alpha = \beta$ becomes:

$$\alpha \approx 1/\gamma \qquad \text{for } \gamma^2 >> 1, \qquad (4)$$

giving the half-angle of the cone in the ultra-relativistic case.

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