RED2013, Relativistic Electrodynamics

Introductory Notes on Tensors -- Solutions (or Hints) to Exercises

Exercise 1: Verify the following simple, but often-used, theorem for arbitrary rank-2 tensors:

$$\sum_{m=1}^{N} \sum_{n=1}^{N} A_{(mn)} B_{[mn]} = 0 .$$

Answer: Because $B_{[mn]}$ is skew-symmetric, its diagonal elements vanish, so terms in the double summation with m & n equal all individually vanish. Because of the symmetry of $A_{(mn)}$ and the skew-symmetry of $B_{[mn]}$, all other terms vanish in pairs: e.g.

$$A_{(12)} B_{[12]} + A_{(21)} B_{[21]} = A_{(12)} B_{[12]} - A_{(12)} B_{[12]} = 0 .$$

Exercise 2: Verify that the components of **ab – ba** are, apart from sign, just the components of $\mathbf{a} \times \mathbf{b}$.

Answer: This must refer to 3 dimensions, because only in that case is the cross product defined. The diagonal components of **ab – ba** are all zero:

$$(ab - ba)_{11} = a_1 b_1 - b_1 a_1 = 0, \text{ etc.}$$

The off diagonal components are:

$$(ab - ba)_{12} = a_1 b_2 - b_1 a_2 = (\mathbf{a} \times \mathbf{b})_3 = -(ab - ba)_{21} ,$$
$$(ab - ba)_{13} = a_1 b_3 - b_1 a_3 = -(\mathbf{a} \times \mathbf{b})_2 = -(ab - ba)_{31} ,$$
$$(ab - ba)_{23} = a_2 b_3 - b_2 a_3 = (\mathbf{a} \times \mathbf{b})_1 = -(ab - ba)_{32} .$$

Exercise 3: Given any two vectors **a** and **b**, verify that their scalar and vector products, **a.b** and $\mathbf{a} \times \mathbf{b}$, can be expressed in cartesian tensor notation as follows:

$$\mathbf{a}.\mathbf{b} = a_r b_r , \qquad \textbf{(i)}$$
$$(\mathbf{a} \times \mathbf{b})_r = \varepsilon_{rst}\, a_s b_t . \qquad \textbf{(ii)}$$

Answer: **(i)** In rectangular cartesian coordinates:

$$\mathbf{a}.\mathbf{b} = a_1 b_1 + a_2 b_2 + \ldots = a_r b_r \qquad \text{(summation convention).}$$

(ii) Eq (ii) has one free index, r. Putting $r = 1, 2, 3$ in turn, and using the properties of the ε tensor:

$$\varepsilon_{1st}\, a_s b_t = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 + \text{zero terms} = a_2 b_3 - a_3 b_2 = (\mathbf{a} \times \mathbf{b})_1 ,$$
$$\varepsilon_{2st}\, a_s b_t = \varepsilon_{231} a_3 b_1 + \varepsilon_{213} a_1 b_3 + \text{zero terms} = a_3 b_1 - a_1 b_3 = (\mathbf{a} \times \mathbf{b})_2 ,$$
$$\varepsilon_{3st}\, a_s b_t = \varepsilon_{312} a_1 b_2 + \varepsilon_{321} a_2 b_1 + \text{zero terms} = a_1 b_2 - a_2 b_1 = (\mathbf{a} \times \mathbf{b})_3 .$$

Exercise 4: Establish the following relationship between the permutation tensor and the Kronecker delta:

$$\varepsilon_{kpq}\,\varepsilon_{krs} = \delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr}\,. \qquad \textbf{(I)}$$

Answer: This is a trifle boring as there are 4 free indices. Each ranges over 1, 2, & 3, so identity **(I)** represents 81 separate equations, which all need to be established. **Hint:** (Holt & Haskell, 1965, referenced in Bibliography, pp 38-9): separate the 81 equations into 6 groups.

Exercise 5: Write the following expressions in cartesian tensor notation:

$$\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \quad \text{and} \quad (\mathbf{a}\times\mathbf{b})\times\mathbf{c}\,.$$

Answer: The dot product $\mathbf{a}\cdot\mathbf{b}$ is $a_m b_m$ in cartesian tensor notation. So the vector

$\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$ has typical (k) component $a_k a_m b_m$.

The cross product has typical (r) component $(\mathbf{a}\times\mathbf{b})_r = \varepsilon_{rst}\,a_s b_t$. So, initially treating $\mathbf{a}\times\mathbf{b}$ as a single vector:

$$[\,(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\,]_k = \varepsilon_{kst}(\mathbf{a}\times\mathbf{b})_s c_t$$

$$= \varepsilon_{kst}(\varepsilon_{spq}\,a_p b_q)c_t \quad \text{(being careful not to double up dummy indices)}$$

So the vector $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ has typical (k) component $\varepsilon_{kst}\varepsilon_{spq}\,a_p b_q c_t$.

Exercise 6: Use cartesian tensor notation to derive the vector identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\,.$$

Answer: The procedure is:

(1) express the LHS in cartesian tensor notation as for $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ in the previous exercise;

(2) manipulate the ε factors so that they have the same (dummy) first index;

(3) apply identity **(I)** in an exercise above; etc.

Exercise 7 (tensor fields): Check the following equivalencies:

$$\nabla V \;\leftrightarrow\; V_{,k} \qquad \textbf{(i)}$$

$$\nabla\cdot\mathbf{A} \;\leftrightarrow\; A_{k,k} \qquad \textbf{(ii)}$$

$$\nabla\times\mathbf{A} \;\leftrightarrow\; \varepsilon_{kpq}A_{q,p} \qquad \textbf{(iii)}$$

Answer:

(i) ∇V has typical (k) component $\partial V/\partial x_k \equiv \partial_k V \equiv V_{,k}$

(ii) $\nabla\cdot\mathbf{A} = \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 \equiv A_{1,1} + A_{2,2} + A_{3,3} \equiv A_{k,k}$

(iii) The RHS has one free index, k. Putting $k = 1, 2, 3$ in turn, and using properties of the ε tensor:

$$\varepsilon_{1pq}A_{q,p} = \varepsilon_{123}A_{3,2} + \varepsilon_{132}A_{2,3} + \text{zero terms} = A_{3,2} - A_{2,3} = (\nabla\times\mathbf{A})_1\,,$$

$$\varepsilon_{2pq}A_{q,p} = \varepsilon_{231}A_{1,3} + \varepsilon_{213}A_{3,1} + \text{zero terms} = A_{1,3} - A_{3,1} = (\nabla\times\mathbf{A})_2\,,$$

$$\varepsilon_{3pq}A_{q,p} = \varepsilon_{312}A_{2,1} + \varepsilon_{321}A_{1,2} + \text{zero terms} = A_{2,1} - A_{1,2} = (\nabla\times\mathbf{A})_3\,.$$

Exercise 8 (with additional wording)**:** Skew rank-2 tensors in 3 dimensions have 3 independent components, which form an *associated vector* defined by:

$$Q_r = \tfrac{1}{2}\varepsilon_{rst} P_{[st]} .$$

Verify that **(a)** inverting this equation to get the tensor in terms of its associated vector yields

$$(P_{[st]}) = \begin{pmatrix} 0 & Q_3 & -Q_2 \\ -Q_3 & 0 & Q_1 \\ Q_2 & -Q_1 & 0 \end{pmatrix},$$

and that **(b)** skew rank-2 tensors in 3D always have determinant zero.

Answer: (a) Putting the free index $r = 1, 2, 3$ in turn:

$$Q_1 = \tfrac{1}{2}\varepsilon_{1st} P_{[st]} = \tfrac{1}{2}\varepsilon_{123}P_{[23]} + \tfrac{1}{2}\varepsilon_{132}P_{[32]} = P_{[23]} ,$$
$$Q_2 = \tfrac{1}{2}\varepsilon_{2st} P_{[st]} = \tfrac{1}{2}\varepsilon_{231}P_{[31]} + \tfrac{1}{2}\varepsilon_{213}P_{[13]} = P_{[31]} ,$$
$$Q_3 = \tfrac{1}{2}\varepsilon_{3st} P_{[st]} = \tfrac{1}{2}\varepsilon_{312}P_{[12]} + \tfrac{1}{2}\varepsilon_{321}P_{[21]} = P_{[12]} .$$

Inverting these equations:

$$P_{[12]} = Q_3 , \quad P_{[13]} = -Q_2 , \quad P_{[23]} = Q_1 , \qquad \text{as required.}$$

(b) Skew rank-2 tensors in 3D always have the structure shown, and hence determinant evaluating to

$$0\,(Q_1{}^2) - Q_3(-Q_1Q_2) - Q_2(Q_1Q_3); \qquad \text{i.e. to zero.}$$

Exercise 9: Show from the tensor transformation rules that *symmetry* ($T_{\beta\alpha} = T_{\alpha\beta}$) and *anti-symmetry* ($T_{\beta\alpha} = -T_{\alpha\beta}$) are invariant properties; *i.e.* independent of the coordinate frame.

Answer: Consider, for example, covariant components of a rank-2 tensor. The tensor transformation rule for these is:

$$T_{\mu'\nu'} = p^{\alpha}{}_{\mu'}\, p^{\beta}{}_{\nu'}\, T_{\alpha\beta} .$$

Re-writing this equation with the free indices μ' and ν' interchanged:

$$\begin{aligned} T_{\nu'\mu'} &= p^{\alpha}{}_{\nu'}\, p^{\beta}{}_{\mu'}\, T_{\alpha\beta} \\ &= p^{\alpha}{}_{\nu'}\, p^{\beta}{}_{\mu'}\, T_{\beta\alpha} && \text{(given } T_{\alpha\beta} = T_{\beta\alpha}\text{)} \\ &= p^{\beta}{}_{\nu'}\, p^{\alpha}{}_{\mu'}\, T_{\alpha\beta} && \text{(interchanging dummy indices } \alpha \text{ and } \beta\text{)} \\ &= p^{\alpha}{}_{\mu'}\, p^{\beta}{}_{\nu'}\, T_{\alpha\beta} && \text{(interchanging order of the two (algebraic) } p \text{ factors)} \\ &= T_{\mu'\nu'} \quad \text{QED} && \text{(by tensor component transformation rule)} \end{aligned}$$

Similarly for anti-symmetry: a –ve sign enters from $T_{\alpha\beta} = -T_{\beta\alpha}$ and carries through.

Exercise 10: Show that any contravariant or covariant (not mixed) r2 tensor, has an invariant decomposition into symmetric & skew parts:

$$T_{\alpha\beta} \equiv \tfrac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) + \tfrac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})$$
$$\equiv T_{(\alpha\beta)} + T_{[\alpha\beta]} \equiv \text{symmetric part} + \text{skew part.}$$

Answer: Consider first a covariant rank-2 tensor. The tensor transformation rule for the components is:

$$T_{\mu'\nu'} = p^{\alpha}{}_{\mu'}\, p^{\beta}{}_{\nu'}\, T_{\alpha\beta}\,.$$

So:
$$T_{\mu'\nu'} = p^{\alpha}{}_{\mu'}\, p^{\beta}{}_{\nu'} \left\{T_{(\alpha\beta)} + T_{[\alpha\beta]}\right\}$$
$$= p^{\alpha}{}_{\mu'}\, p^{\beta}{}_{\nu'}\, T_{(\alpha\beta)} + p^{\alpha}{}_{\mu'}\, p^{\beta}{}_{\nu'}\, T_{[\alpha\beta]} \qquad (1)$$
$$= \text{transform of } T_{(\alpha\beta)} + \text{transform of } T_{[\alpha\beta]}\,.$$

That is, linearity of the tensor transform operation implies that the transform of the sum is the sum of the transforms.

Also: interchanging the dummy indices α & β gives

$$p^{\alpha}{}_{\mu'}\, p^{\beta}{}_{\nu'}\, T_{(\alpha\beta)} = p^{\beta}{}_{\mu'}\, p^{\alpha}{}_{\nu'}\, T_{(\beta\alpha)} = p^{\alpha}{}_{\nu'}\, p^{\beta}{}_{\mu'}\, T_{(\alpha\beta)}\,,$$

after interchanging the order of the (algebraic) p factors and using the symmetry of $T_{(\alpha\beta)}$. This result states that the transform of $T_{(\alpha\beta)}$ in Eq (1) is symmetric in μ' & ν'. In the same way, it follows that the transform of $T_{[\alpha\beta]}$ in Eq (1) is skew symmetric in μ & ν. So we can write Eq (1) as:

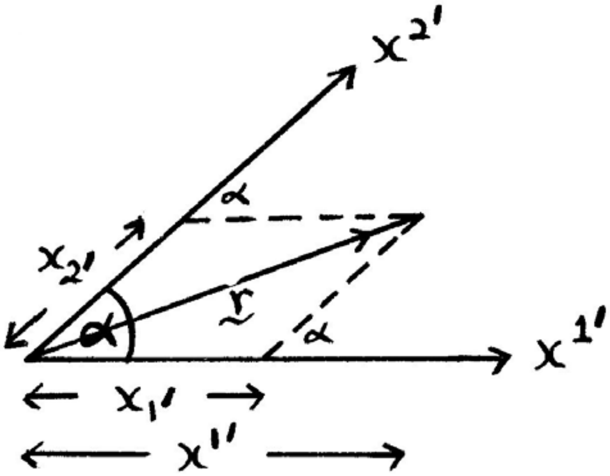
$$T_{\mu'\nu'} = T_{(\mu'\nu')} + T_{[\mu'\nu']}\,,$$

And the decomposition survives the transformation of coordinates.

The same result for a contravariant rank-2 tensor follows from exactly the same steps.

Exercise 11: Show that the covariant components $x_{1'}$ & $x_{2'}$ of $\mathbf{r}$ in the oblique frame are given by parallel projection, as shown in the diagram.

Answer: See problem Q1(a) and solution below.



Exercise 12: Show that for cartesian tensors there is no distinction between contravariance and covariance.

Answer: The inverse pair of linear coordinate transformations

$$x^{m'} = a_n{}^m x^n + a^m \qquad \& \qquad x^n = b_m{}^n x^{m'} + b^n , \tag{1}$$

are said to be *orthogonal* if $b_m{}^n = a_n{}^m$. In matrix notation, the transformation

$\mathbf{x}' = \alpha\mathbf{x} + \mathbf{a}$, with matrix α and column vectors $\mathbf{x}'$, $\mathbf{x}$ & $\mathbf{a}$,

has inverse

$\mathbf{x} = \beta\mathbf{x}' + \mathbf{b}$, where $\beta \equiv \alpha^{-1}$ & $\mathbf{b} \equiv -\alpha^{-1}\mathbf{a}$.

The given condition $b_m{}^n = a_n{}^m$ is

$\beta = \alpha^{\mathrm{T}}$; *i.e.* $\alpha^{-1} = \alpha^{\mathrm{T}}$ or $\alpha\alpha^{\mathrm{T}} = \mathbf{I}$ (the unit matrix):

the transformation matrix is *orthogonal: its inverse is equal to its transpose*.

We are going to show that under orthogonal transformations (*i.e.* for cartesian tensors) there is no difference between contravariance and covariance.

The inverse pair of linear coordinate transformations in (1) above yield the following tensor transformation coefficients:

$$p^{m'}{}_n \equiv \partial x^{m'}/\partial x^n = a_n{}^m \qquad \& \qquad p^n{}_{m'} \equiv \partial x^n/\partial x^{m'} = b_m{}^n .$$

Hence vector components transform according to

$$A^{m'} = p^{m'}{}_n A^n = a_n{}^m A^n \qquad \& \qquad A_{m'} = p^n{}_{m'} A_n = b_m{}^n A_n . \tag{2a,b}$$

If we now impose the orthogonality condition $b_m{}^n = a_n{}^m$, Eq (2b) becomes

$$A_{m'} = a_n{}^m A_n . \tag{3}$$

Comparison of Eqs (2a) & (3) shows that contravariant and covariant components transform in the same way and so are indistinguishable.

Note: We can readily show as follows that in Euclidean space referred to rectangular cartesian coordinates, there is no difference between contravariance and covariance:

In Euclidean space with rectangular cartesian coordinates, the metric tensor is the Kronecker delta: $g_{mn} = \delta_{mn}$. So, for a tensor $(A^m{}_n)$ for example, it follows that

$$A_{mn} = g_{mk} A^k{}_n = \delta_{mk} A^k{}_n = A^m{}_n ,$$

and similarly for any tensor indices -- raising and lowering of indices makes no difference, so there is no distinction between contravariance and covariance.

Exercise 13 (with additional wording)**:**

Given a contravariant vector (A^ν), we can define its covariant components by $A_\nu = g_{\nu\mu}A^\mu$.

Given a covariant vector (A_ν), we can define its contravariant components by $A^\nu = g^{\nu\mu}A_\mu$.

From these relations, deduce the fundamental tensor for oblique coordinates in 2 dimensions (see Section 4.3 in the *Introductory Notes on Tensors*).

Answer: See problem Q1(b) and solution below.

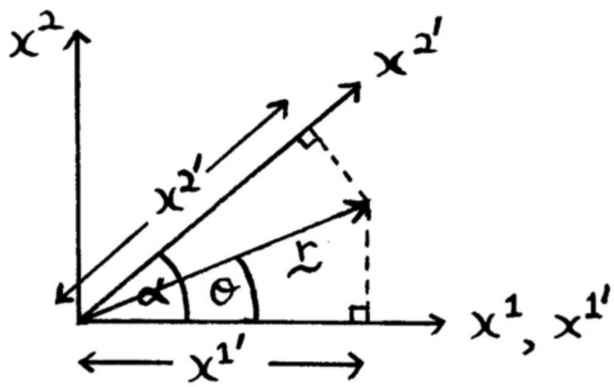
Q1. Contravariance & Covariance. In a 2-dimensional Euclidean space, a vector **V** has components $V^1 = V_1$ and $V^2 = V_2$ relative to orthogonal cartesian coordinates x^1 and x^2. Oblique axes x'^1 and x'^2, with $x'^1 \equiv x^1$, are formed by rotating the x^2 axis toward the x^1 axis by an angle $\pi/2 - \alpha$.

(a) Show that the contravariant and covariant components of **V** in the oblique frame are given respectively by perpendicular projection onto the axes and by projection parallel to the axes.

(b) What are the components of the metric tensor, in both covariant and contravariant forms, in the new frame?

(c) Show that the contravariant and covariant components of **V** are inter-related by the metric tensor in the standard way.

Solution of Q1: To study the covariant and contravariant transformations, our starting point has to be the forward and reverse coordinate transformations. From the transformation equations, the second step follows: forming the two sets of partial derivatives -- of the 'new' coordinates with respect to the 'old' and of the 'old' coordinates with respect to the 'new'.



The transformation from the orthogonal to the oblique coordinates is (see the diagram)

$$x^1 = x^{1'} \quad \&$$

$$x^{2'} = r\cos(\alpha - \theta) = x^1\cos\alpha + x^2\sin\alpha$$

which inverts algebraically to

$$x^1 = x^{1'} \quad \&$$

$$x^2 = -x^{1'}\cot\alpha + x^{2'}\operatorname{cosec}\alpha\ .$$

Now that we have the forward and reverse coordinate transformations, we can form the sets of partial derivatives:

$$p^{m'}{}_{n} \equiv \partial x^{m'}/\partial x^{n} \quad \& \quad p^{n}{}_{m'} \equiv \partial x^{n}/\partial x^{m'}\ .$$

Thus, taking the 1st & 2nd indices to label rows & columns, respectively:

$$(p^{m'}{}_{n}) = \begin{pmatrix} 1 & 0 \\ \cos\alpha & \sin\alpha \end{pmatrix} \qquad \text{(I)}$$

and

$$(p^{n}{}_{m'}) = \begin{pmatrix} 1 & 0 \\ -\cot\alpha & \csc\alpha \end{pmatrix}. \qquad \text{(II)}$$

Note that these arrays, considered as matrices, are mutually inverse.

(a) First, **contravariant components.** The transformation rule $V^{m'} = (\partial x^{m'}/\partial x^n)V^n$ for the contravariant components of a vector gives, using Eq (I):

$$V^1 = V^{1'} \qquad \& \qquad V^{2'} = V^1\cos\alpha + V^2\sin\alpha\,, \qquad (1a, b)$$

for the contravariant components of **V** in terms of its components V^1 & V^2 (which can also be written as V_1 & V_2) in the original rectangular coordinates.

Equations (1a,b) can also obtained trigonometrically by perpendicular projection onto the axes, as shown above (see first diagram) for the displacement vector **r**: the same trigonometry applies to projections of **V**.

Second: **covariant components.** The transformation rule $V_{m'} = (\partial x^n/\partial x^{m'})V_n$ for the covariant components of a vector gives, using Eq (II):

$$V_{1'} = V_1 - V_2\cot\alpha \qquad \& \qquad V_{2'} = V_2\operatorname{cosec}\alpha\,. \qquad (2a, b)$$

We need to see if these are obtained trigonometrically by parallel projection onto the axes.

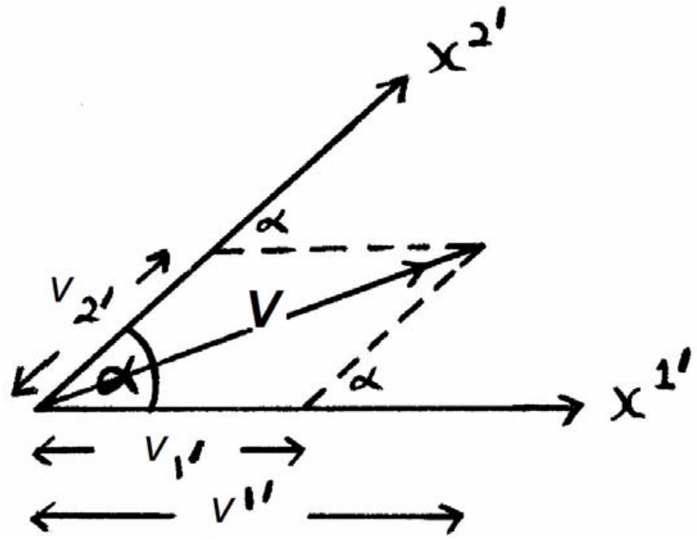
From the 2nd diagram above (resolving the two dashed lines onto the oblique axes), we see that

$$V_{1'} + V_{2'}\cos\alpha = V^1 \qquad \& \qquad V_{2'} + V_{1'}\cos\alpha = V^1\cos\alpha + V^2\sin\alpha\,, \qquad (3a, b)$$

where we have replaced $V^{1'}$ and $V^{2'}$ on the RHSs by using Eqs (1). Solving Eqs(3) for $V_{1'}$ and $V_{2'}$;

$$V_{1'} = V^1 - V^2\cot\alpha \qquad \& \qquad V_{2'} = V^2\operatorname{cosec}\alpha\,, \qquad (4a, b)$$

which are the same as Eqs (2) given that there is no distinction between contravariant and covariant components in the undashed, rectangular, frame.



(b) In a 2-dimensional space referred to rectangular cartesian axes, the metric tensor has the form $(g_{mn}) = \text{diag}\{1, 1\}$. Its components in the oblique frame are given by the transformation rule for covariant components of a 2nd-rank tensor:

$$g_{m'n'} = p^r{}_{m'}p^s{}_{n'}g_{rs}\,.$$

Hence, using $g_{12} = 0 = g_{21}$ and (II):

$$g_{1'1'} = p^r{}_{1'}p^s{}_{1'}g_{rs} = (p^1{}_{1'})^2 g_{11} + (p^2{}_{1'})^2 g_{22} = 1 + \cot^2\alpha = \operatorname{cosec}^2\alpha,$$

$$g_{1'2'} = p^r{}_{1'}p^s{}_{2'}g_{rs} = p^1{}_{1'}p^1{}_{2'}g_{11} + p^2{}_{1'}p^2{}_{2'}g_{22} = 0 - (\cot\alpha)(\operatorname{cosec}\alpha) = -(\cos\alpha)(\operatorname{cosec}^2\alpha),$$

$$g_{2'1'} = g_{1'2'} \qquad \text{(symmetry is an invariant property)}$$

$$g_{2'2'} = p^r{}_{2'}p^s{}_{2'}g_{rs} = (p^1{}_{2'})^2 g_{11} + (p^2{}_{2'})^2 g_{22} = \operatorname{cosec}^2\alpha\,.$$

That is:

$$(g_{m'n'}) = \csc^2\alpha \begin{pmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{pmatrix}.$$

Note: the metric tensor is no longer diagonal. For the metric tensor, the contravariant components are, by definition, given by taking the inverse matrix of $(g_{m'n'})$:

$$(g^{m'n'}) = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}.$$

A check: the form diag{1, 1} is regained when $\alpha = \pi/2$.

(c) Lowering vector indices in the oblique frame, using the metric tensor components just calculated:

$$\begin{aligned} V'_{1'} &= g_{1'n'}V'^{n} = g_{1'1'}V'^{1'} + g_{1'2'}V'^{2'} = \operatorname{cosec}^2\alpha\,(V'^{1'} - \cos\alpha V'^{2'}) \\ &= \operatorname{cosec}^2\alpha[V^1 - \cos\alpha(V^1\cos\alpha + V^2\sin\alpha)] && \text{by (1)} \\ &= V^1 - V^2\cot\alpha = V_1 - V_2\cot\alpha && \text{in agreement with (2) ;} \end{aligned}$$

the equivalence of contravariant & covariant components in the undashed (orthogonal) frame has been used. Also:

$$\begin{aligned} V'_2 &= g_{2'n'}V'^{n} = g_{2'1'}V'^1 + g_{2'2'}V'^2 = \operatorname{cosec}^2\alpha(-\cos\alpha V'^1 + V'^2) \\ &= \operatorname{cosec}^2\alpha[-\cos\alpha V^1 + (V^1\cos\alpha + V^2\sin\alpha)] && \text{by (1)} \\ &= V^2\operatorname{cosec}\alpha = V_2\operatorname{cosec}\alpha && \text{in agreement with (2) .} \end{aligned}$$

Raising vector indices in the oblique frame:

$$\begin{aligned} V'^{1'} &= g^{1'n'}V_{n'} = g^{1'1'}V_{1'} + g^{1'2'}V_{2'} = V'_1 + \cos\alpha V'_2 \\ &= (V_1 - V_2\cot\alpha) + \cos\alpha(V_2\operatorname{cosec}\alpha) && \text{by (2)} \\ &= V_1 = V^1 && \text{in agreement with (1) ;} \end{aligned}$$

and

$$\begin{aligned} V'^{2'} &= g^{2'n'}V_{n'} = g^{2'1'}V_{1'} + g^{2'2'}V_{2'} = \cos\alpha V_{1'} + V_{2'} \\ &= \cos\alpha(V_1 - V_2\cot\alpha) + V_2\operatorname{cosec}\alpha && \text{by (2)} \\ &= V_1\cos\alpha + V_2\sin\alpha = V^1\cos\alpha + V^2\sin\alpha, && \text{in agreement with (1) .} \end{aligned}$$

That is, lowering and raising indices reproduces the relations:

$$V^{1'} = V^1 \quad \& \quad V^{2'} = V^1\cos\alpha + V^2\sin\alpha, \qquad (1)$$

$$V_{1'} = V_1 - V_2\cot\alpha \quad \& \quad V_{2'} = V_2\operatorname{cosec}\alpha, \qquad (2)$$

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