University of Western Australia
Australian National University

Physics Honours & Masters

Relativistic Electrodynamics

Notes on

Maxwell's Equations and Relativity

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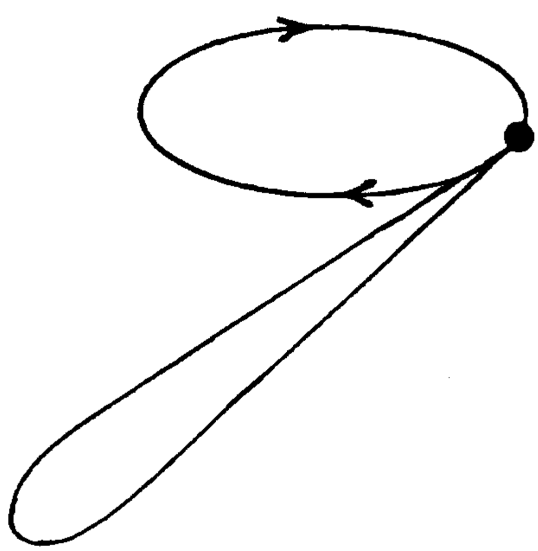

Maxwell's Equations and Relativity

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01 Inertial Frames & Lorentz Transformations

An event has space-time coordinates (ct, x, y, z) in inertial frame $\mathcal{S}$ and (ct', x', y', z') in $\mathcal{S}\,'$. We recall the Lorentz transformation (LT) between $\mathcal{S}$ & $\mathcal{S}\,'$ in *standard configuration*, meaning that the origins coincide at $t = 0 = t'$ and the frames' relative motion is along the common x-x' axes:

$$ct' = \gamma(v)(ct - vx/c)\ ,\ x' = \gamma(v)(x - vt)\ ,$$

$$y' = y\ ,\ z' = z\ , \tag{1}$$

with c the vacuum speed of light.

The *Lorentz factor* is

$$\gamma(v) \equiv 1/\sqrt{1 - \beta^2}\ , \qquad \beta \equiv v/c\ . \tag{2}$$

For standard configuration, the y & z coordinates are unaffected; only the x & ct axes transform. Note: c is from Latin, *celeritas*, swiftness, celerity.

Consider an event $(ct\,,\,\mathbf{r}\,)$ in $\mathcal{S}$, at position 3-vector $\mathbf{r} = (x, y, z)$. Let the velocity of $\mathcal{S}\,'$ with respect to $\mathcal{S}$ be $\mathbf{v}$ in any direction, but choose the $\mathcal{S}$ & $\mathcal{S}'$ axes to be parallel.

For the coordinates $(ct'\,,\,\mathbf{r}')$ of the event in $\mathcal{S}\,'$ in matrix notation with $\mathbf{r}$ & $\mathbf{v}$ as column vectors and $\mathbf{v}^{\mathrm{T}}$ a row vector:

$$\begin{pmatrix} ct' \\ \mathbf{r}' \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & -\mathbf{v}^{\,\mathrm{T}}/c \\ -\mathbf{v}/c & \mathrm{P_v} + (\mathrm{I} - \mathrm{P_v})/\gamma(v) \end{pmatrix} \begin{pmatrix} ct \\ \mathbf{r} \end{pmatrix} . \quad (3)$$

Here I is the unit matrix, $\mathrm{P_v} = \mathbf{v}\mathbf{v}/v^2$ projects onto $\mathbf{v}$, and $\mathrm{I} - \mathrm{P_v}$ denotes projection transverse to $\mathbf{v}$.

Reading: Jackson's Sect. 11.3, *Lorentz Transformations and Basic Kinematic Results of Special Relativity*, pp 524–30.

02 Coordinate Transformations & Metric

The spacetime of special relativity is described by the *Minkowski metric tensor*, $(\eta_{\mu\nu})$, with components shown in

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4}$$

valid in any inertial reference frame. This, as a matrix, is equal to its inverse, $\eta^{\mu\nu}$, in these frames.

Transformations between inertial frames can be represented by the *Lorentz transformation matrix* Λ: for standard configuration,

$$\left(\Lambda^{\mu'}{}_{\nu}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

where μ' indicates a row and ν a column.

This is the matrix of a *Lorentz boost* (velocity change without spatial rotation or translation), mathematically similar to a space-time rotation of the x and ct coordinates.

The signs in this matrix depend on the *signature* chosen for the metric tensor. Here we use $(+,-,-,-)$, corresponding to the metric tensor above, which can be written:

$$(\eta^{\mu\nu}) = \mathrm{diag}\{1,-1,-1,-1\}\,. \tag{6}$$

A transformation from one inertial frame to another follows the rule:

$$\eta_{\alpha\beta} = \Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}\eta_{\mu'\nu'}\,, \tag{7}$$

with implied summation from 0 to 3 over the repeated indices (μ' & ν') – the *Einstein summation convention*.

More generally, 3D axis rotations can be included in Λ. The *Poincaré group* is the most general group of transformations that preserves the Minkowski metric; it is the *physical symmetry underlying special relativity*.

In tensor form, the equations of physics are manifestly invariant under the Poincaré group, and so automatically consistent with special relativity. We also find that apparently unrelated equations (*e.g.* those for energy and momentum conservation) become connected when formed into one tensor equation.

A single quantity invariant under Lorentz transformations (LTs) is a *Lorentz scalar* or *4-scalar* (0-rank 4-tensor).

The differential separation 4-vector, with components dx^μ, between neighbouring events has squared magnitude:

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}\, dx^\mu\, dx^\nu \\ &= c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \,. \end{aligned} \tag{8}$$

This is a Lorentz invariant, taking the same value in all inertial frames; ds is the *line element*.

If ds^2 is +ve, then $d\tau = ds/c$ is the differential *proper time* separation. If ds^2 is −ve, then $\sqrt{-ds^2}$ is the differential *proper distance* between the events.

All physical quantities can be represented as tensors or tensor components. To transform from one frame to another, the standard tensor transformation rule applies:

$$T^{i'_1,i'_2,\ldots,i'_p}_{j'_1,j'_2,\ldots,j'_q} =$$

$$\Lambda^{i'_1}{}_{i_1}\Lambda^{i'_2}{}_{i_2}\cdots\Lambda^{i'_p}{}_{i_p}\Lambda_{j'_1}{}^{j_1}\Lambda_{j'_2}{}^{j_2}\cdots\Lambda_{j'_q}{}^{j_q}T^{i_1,i_2,\ldots,i_p}_{j_1,j_2,\ldots,j_q}\,, \tag{9}$$

where $\Lambda_{j'_k}{}^{j_k}$ is the inverse matrix of $\Lambda^{j'_k}{}_{j_k}$.

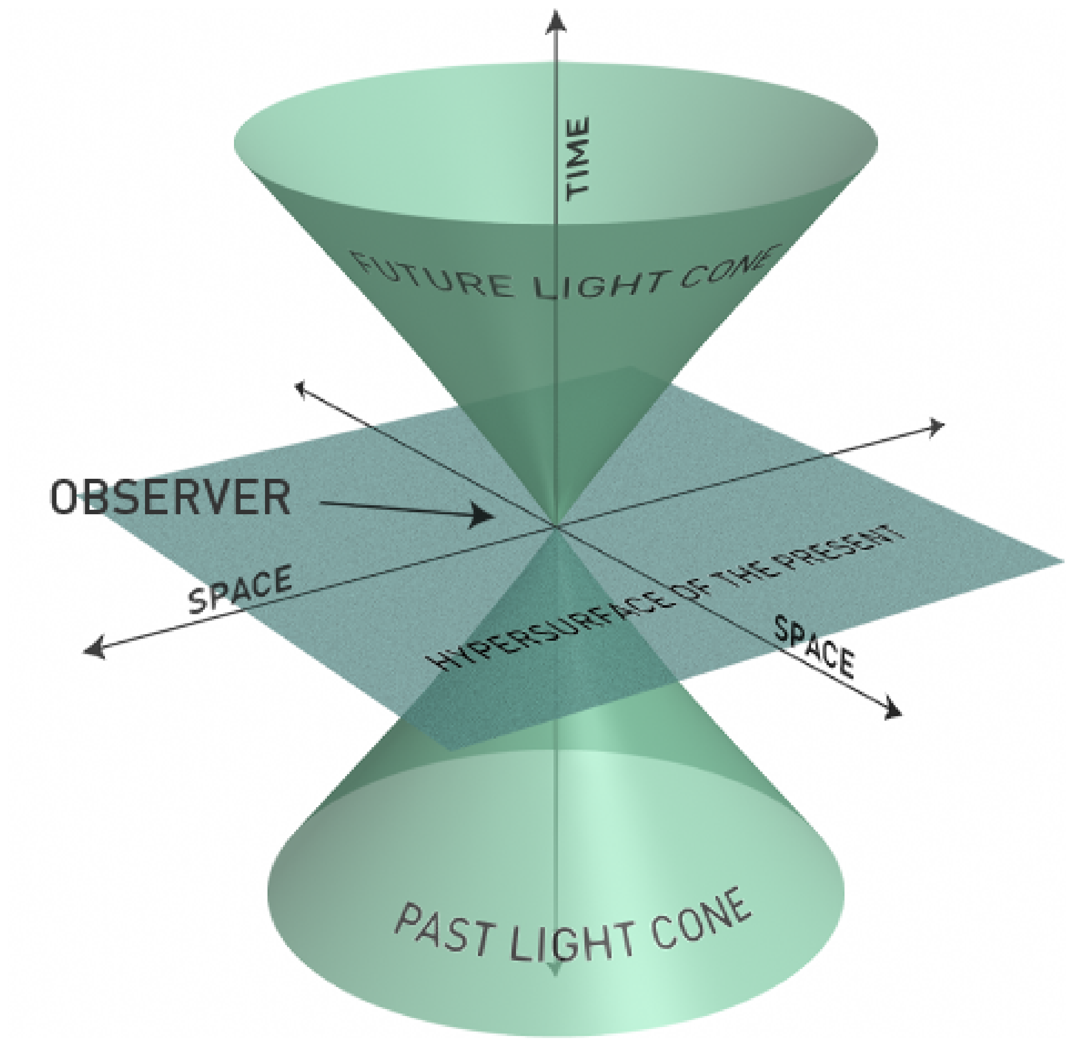
To illustrate, we transform the position of an event from frame $\mathcal{S}$ to frame $\mathcal{S}\,'$ by calculating:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \left(x^{\mu'}\right) = \left(\Lambda^{\mu'}{}_{\nu}x^{\nu}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(ct-\beta x) \\ \gamma(x-\beta ct) \\ y \\ z \end{pmatrix}, \tag{10}$$

which is the usual LT.

Reading: Jackson's Sect. 11.6, *Mathematical Properties of the Space-Time of Special Relativity*, pp 539–43.



The Light Cone, by stib.

03 The Lorentz and Poincaré Groups

Our notes for this topic are:

Jackson's Sect. 11.7, *Matrix Representations of Lorentz Transformations, Infinitesimal Generators*, pp 543–8;

Jackson's Sect. 6.10, *Transformation Properties of Electromagnetic Fields and Sources Under Rotations, Spatial Reflections, and Time Reversal*, pp 267–73.

Note in particular:

• The *Lorentz group* consists of *spatial rotations* (a non-Abelian subgroup) together with *Lorentz boosts*.

• *Boosts alone do not form a group*: the composition of two non-collinear boosts is not a boost but a boost plus a spatial rotation.

• Representation of LTs by non-commuting matrices shows that *the result of successive LTs depends on their order*.

• ***Proper LTs*** **are ones continuously connected with the identity transformation, and so must have det $\Lambda = +1$.**

• ***Improper LTs*, meaning ones not continuously connected with the identity, may have det $\Lambda = -1$ (*e.g.* space inversion) or det $\Lambda = +1$ (*e.g.* combined space & time inversion).**

• **The *Poincaré group* contains, in addition, space-time translations (an Abelian subgroup) – it is the *full symmetry group of special relativity*.**

Note: Although generally attributed to Minkowski, the 4-dimensional interpretation of special relativity was introduced by Poincaré: see

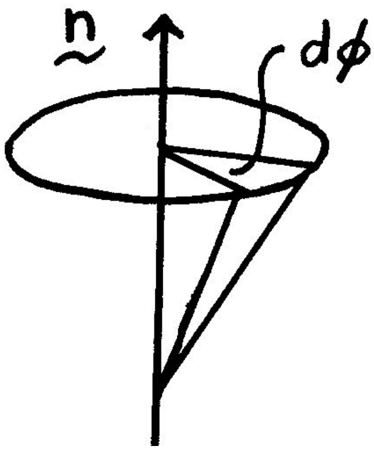
Poincaré, H. (1906), Sur la dynamique de l'électron. *Rendiconti del Circolo matematico di Palermo*, tome 21, pages 129–76; reprinted in his Œuvres, vol. 9, pp 494–550 (see pp 541–2 etc.).

04 Transformations to Rotating Frames

An infinitesimal rotation (but *not* a finite one) can be represented by a vector, meaning that infinitesimal rotations add as vectors:

$$\mathbf{d}\boldsymbol{\phi} = \mathbf{n}\, d\phi \tag{11}$$

with $\mathbf{n}$ the unit vector along the rotation axis and $d\phi$ the angular displacement.



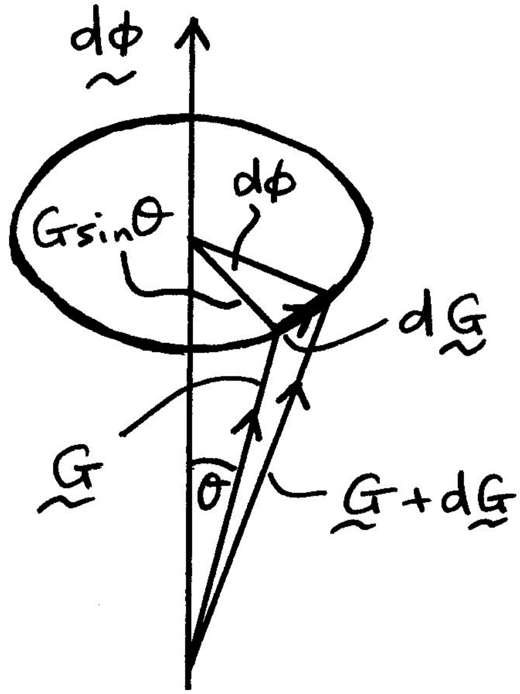
Infinitesimal rotation vector $\mathbf{n}d\phi$.

This section deals with a form of relativity at the Newtonian level: transformations between frames with relative speeds well below c. One of the frames is non-inertial.

We obtain a result that we'll need soon in a context that involves special relativity, *even though the speeds are non-relativistic*.

Let an arbitrary vector G be rotated by an infinitesimal rotation $\mathbf{d}\boldsymbol{\phi}$ to become $\mathbf{G} + \mathbf{dG}$, as in the above diagram. The second diagram shows that the magnitude $dG = (G \sin\theta)\, d\phi$ and that dG is in the direction of $\mathbf{d}\boldsymbol{\phi} \times \mathbf{G}$; so:

$$\mathbf{dG} = \mathbf{d}\boldsymbol{\phi} \times \mathbf{G}\,. \tag{12}$$



Vector in a rotating frame.

The time rate of change of a vector fixed in a body that is rotating with angular velocity $\boldsymbol{\omega}$ follows on dividing this equation through by dt. With $\boldsymbol{\omega} \equiv \mathbf{d}\boldsymbol{\phi}\, dt$:

$$\mathbf{dG}/dt = \boldsymbol{\omega} \times \mathbf{G}\,. \tag{13}$$

We introduce two reference frames:

- **an inertial frame, or *space frame*, I, 'fixed in space';**
- **a rotating frame, or *body frame*, R, fixed in the body.**

The eqn for $\mathbf{dG}/dt$ holds in I, for $\mathbf{G}$ fixed in R.

We allow $\mathbf{G}$ to change *as seen in the body frame* R, at rate $(\mathbf{dG}/dt)_{\mathrm{R}}$. Its rate of change in the inertial frame will be the sum of the two rates:

$$\left(\frac{\mathbf{dG}}{dt}\right)_{\mathbf{I}} = \left(\frac{\mathbf{dG}}{dt}\right)_{\mathbf{R}} + \boldsymbol{\omega}_{\mathbf{I}} \times \mathbf{G} \tag{14}$$

i.e. **rate in R + rate owing to rotation with R.**

We have dG/dt as the sum of an intrinsic rate and a rate due to being dragged around by the rotating body.

As G is arbitrary, this is a relation between time derivatives in R and those in I. So we abstract out an operator relation:

$$\left(\frac{d}{dt}\right)_{\mathbf{I}} = \left(\frac{d}{dt}\right)_{\mathbf{R}} + \boldsymbol{\omega}_{\mathbf{I}} \times \quad . \tag{15}$$

This operator result enables us *to transform time derivatives of vectors between inertial and rotating frames*. In this way, we can treat motion as seen from a rotating frame, such as one fixed to the surface of the Earth.

Applying the relation to a velocity vector relates the accelerations as seen in the two frames. The acceleration in the rotating frame contains the centrifugal and Coriolis effects.

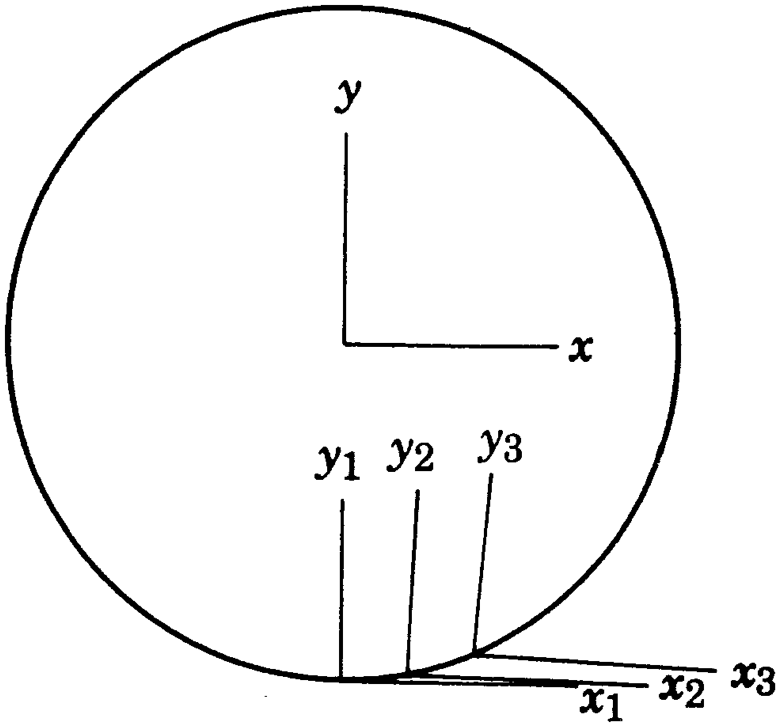
05 Thomas Precession

Our notes for this topic are Jackson's Sect. 11.8, *Thomas Precession*, pp 548–53.

Note in particular:

- ***Thomas precession*** **is a direct outcome of the fact that *the composition of two non-collinear Lorentz boosts is not a boost but a boost plus a spatial rotation*. For an object in an orbit,** *e.g.* **an electron in an atom, integration of infinitesimal rotations around the orbit leads to a precession.**

- **Thomas precession in atoms is an important effect of special relativity, even though the speeds concerned are much less than c.**



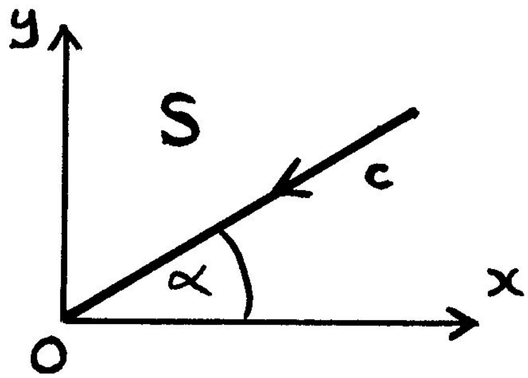
Thomas precession of inertial frames.

The above figure is from:

R. Eisberg & R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles*, Wiley, 2nd ed., 1985; Appendix O, *The Thomas Precession*, pp O-1 to O-3.

06 Aberration of Light

The speed of light is independent of the observer's motion but its velocity is not – *the direction of a light ray depends on the motion of the observer*, an effect called the *aberration of light*. It is analogous to the slanting toward you of vertically falling rain if you run through it.



An incoming light ray.

Let inertial frames $\mathcal{S}$ and $\mathcal{S}'$ be in standard configuration, with v the speed of $\mathcal{S}'$ with respect to $\mathcal{S}$. An object's 3-velocities as seen in these frames, $\mathbf{u}$ and $\mathbf{u}'$ (speeds u & u'), are related by the velocity transformation rule:

$$\mathbf{u}' = \frac{(\, u_x - v \,,\, u_y/\gamma(v) \,,\, u_z/\gamma(v) \,)}{1 - u_x v/c^2}\,. \tag{16}$$

We apply this equation to an incoming light ray travelling in the x-y plane at angle α to the x axis, or α' as seen in $\mathcal{S}'$, observed at the origins of $\mathcal{S}$ & $\mathcal{S}'$. We use β for v/c and γ for γ(v).

Our 'object' is a photon moving in the x-y plane with $u = c = u'$ (speed c in both frames):

$$\mathbf{u} = -c\,(\,\cos\alpha\,,\,\sin\alpha\,)\,, \tag{17}$$

$$\mathbf{u}' = -c\,(\,\cos\alpha'\,,\,\sin\alpha'\,)\,. \tag{18}$$

Substituting this pair into the previous equation gives:

$$(\,\cos\alpha'\,,\,\sin\alpha'\,) = \frac{(\cos\alpha + \beta\,,\,\gamma^{-1}\sin\alpha)}{1 + \beta\cos\alpha}\,. \tag{19}$$

Taking the second of this equation pair and writing α' as $\alpha - \Delta\alpha$ gives:

$$\sin(\alpha - \Delta\alpha) \approx (\sin\alpha)(1 - \beta\cos\alpha) \quad \textbf{for } \beta << 1\,. \tag{20}$$

But, from trigonometry:

$$\sin(\alpha - \Delta\alpha) \approx \sin\alpha - (\cos\alpha)\Delta\alpha \quad \textbf{for } |\Delta\alpha| << 1\,. \tag{21}$$

Comparison of these two formulas shows that:

$$\Delta\alpha \approx \beta \sin\alpha \quad \textbf{for } \beta << 1 \,. \tag{22}$$

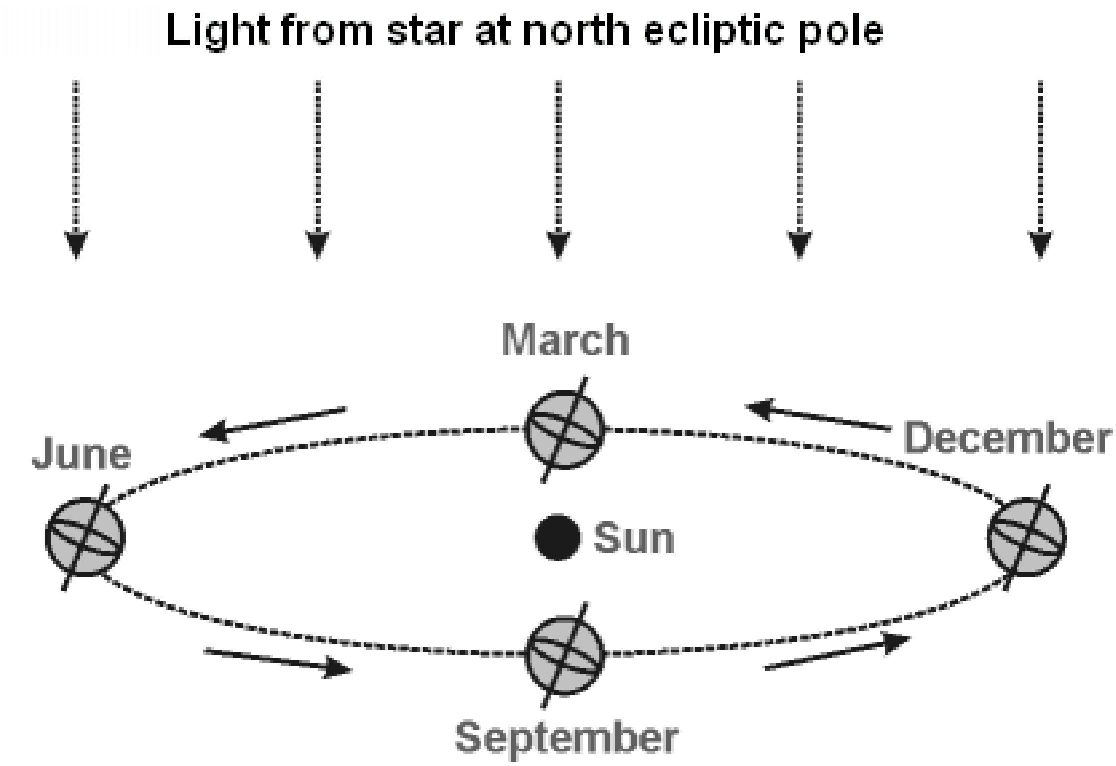
As might be expected, the effect vanishes for $\alpha = 0$ (a head-on ray) and is a maximum for $\alpha = \pi/2$ (ray transverse to the direction of relative motion).

Aberration is a 'semi-relativistic effect' in that it is first-order in β as distinct from more strictly relativistic effects such as mass increase, length contraction and time dilation, which depend on γ and hence β^2.

Let v_{E} denote the Earth's orbital speed of 30 km/s, and $\beta_{\mathrm{E}} \equiv v_{\mathrm{E}}/c = 1 \times 10^{-4}$. As 1 radian $= 2 \times 10^5$ arcsec:

$$\max(\Delta\alpha) \approx \beta_{\mathrm{E}} \approx 20 \textbf{ arcsec} \,, \tag{23}$$

called the *constant of aberration*.



Stellar aberration, after Portnadler.

As the Earth swings around its solar orbit, *a star appears to trace out an ellipse around its non-aberrated position.* The ellipses have semi-major axes of 20 arcsec parallel to the ecliptic; they close down to straight lines for ecliptic stars, and open out to circles at the poles of the ecliptic.

This effect, *annual aberration*, was discovered by Bradley and Molyneux in the 1720s, using a vertical telescope to observe γ Draconis as it passed almost overhead. Bradley was looking for *stellar parallax*, due to finite stellar distances and Earth's orbital motion, but realized that the observed effect had the wrong phase for parallax.

As well as being the discovery of a new and unexpected physical effect, this was also *the first direct detection of the Earth's motion around the Sun*.

In addition, Bradley deduced that light takes 493 s, with an uncertainty of 5–10 s, to cover 1 AU. The modern value is 499 s.

See:

Bradley, J. (1728), A Letter from the Reverend Mr. James Bradley Savilian Professor of Astronomy at Oxford, and F.R.S. to Dr Edmond Halley Astronom. Reg. &c. giving an Account of a new discovered Motion of the Fix'd Stars, *Phil. Trans. R. Soc. (London)*, v 35, pp 637–661. **Some sources give p 660 as the final page, but that omits Bradley's postscript on p 661.**

Stellar Parallax

Unlike aberration, *stellar parallax* is distance-dependent. Because the stars are so far away, it is much smaller than aberration, 762 mas (milliarcsecond) in amplitude for the nearest star to the Sun (Proxima Centauri). It was detected in the 1820s by Henderson (α Cen, 745 mas), Bessel (61 Cyg, 294 mas) and Struve (Vega, 129 mas).

A star's distance, d, in parsecs (from 'parallax seconds') is the inverse of its parallax, often denoted by π, in arcseconds. For α Cen, $\pi = 745$ mas gives $d = 1.33$ pc. [A *parsec (pc)* is the distance at which 1 astronomical unit (AU) subtends an angle of 1 arcsec; 1 pc = 3.26 light years.]

For further diagrams on aberration and stellar parallax, see: *Introductory Notes on Relativistic Electrodynamics.*

07 Global Aberration

The two aberration equations

$$(\cos\alpha', \sin\alpha') = \frac{(\cos\alpha + \beta, \gamma^{-1}\sin\alpha)}{1 + \beta\cos\alpha}, \qquad (24)$$

each express α' as a function of α with parameter β. They can be combined as follows to 'separate the variables', expressing α' as the product of a function of β with a function of α.

The method uses the trigonometric identity:

$$\tan\left(\frac{\theta}{2}\right) \equiv \frac{\sin\theta}{1 + \cos\theta}. \qquad (25)$$

Putting $\theta = \alpha'$ in this identity and using the aberration equations for $\sin\alpha'$ and $\cos\alpha'$:

$$\begin{aligned} \tan(\alpha'/2) &= \frac{\gamma^{-1}\sin\alpha}{(1 + \beta\cos\alpha) + (\cos\alpha + \beta)} \\ &= \frac{\gamma^{-1}\sin\alpha}{(1 + \beta)(1 + \cos\alpha)}. \end{aligned} \qquad (26)$$

Using the trig. identity again, but with α for θ, yields the 'separated' form of aberration equation:

$$\tan(\alpha'/2) = B(\beta)\tan(\alpha/2) \tag{27}$$

with

$$B(\beta) \equiv \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \tag{28}$$

This result describes fully relativistic aberration, referred to here as *global aberration*. It is generally called 'relativistic aberration', but all aberration is to do with relative motion of the emitter and the observer.

Rays with $\alpha = \pi/2$ separate the 'forward' and 'backward' halves of the Universe. The last two equations show that a moving observer sees the forward half Universe contracted by aberration into a cone of half-angle

$$\Lambda(\beta) \equiv 2\,\arctan[B(\beta)]\,. \tag{29}$$

Special cases:

- **for** $\beta << 1$**,** $B(\beta) \approx 1 - \beta$ **and** $\Lambda \approx (\pi/2) - \beta$
- **for** $\beta = 1/2, \Lambda = 60^{\text{o}}$
- **for** $\beta = 0.99, \Lambda = 8.1^{\text{o}}$
- **for** $\beta \to 1, \gamma^{-2} \equiv 1 - \beta^2 \approx 2(1-\beta)$ **so** $B(\beta) \approx 1/(2\gamma)$**; hence** $\Lambda \to 1/\gamma$ **for extremely relativistic motion**
- **a 1-GeV electron has** $\gamma = E/(m_0 c^2) = 2000$**; so** $\Lambda = 1/2000$ **radians = 1.7 arcmin**

The last point illustrates the

- ***extreme sharpening of the beam radiated by highly relativistic electrons***

known as the *searchlight effect* (or headlight effect), and tells us why

- ***aberration is a key ingredient in the nature of synchrotron radiation*.**

08 Spacetime Vectors

The spacetime location of an event is expressed by

$$(R^\mu) = \left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z) = (ct, \mathbf{r}) \qquad (30)$$

where $\mathbf{r} \equiv (x, y, z)$ and $x^0 \equiv ct$ is the time coordinate with dimension of distance, as for the spatial dimensions.

Superscripts will denote *contravariant indices* ranging over 0 to 3, rather than exponents, except when indicating a square. Subscripts (also 0 to 3) are *covariant indices*.

The *prototype contravariant 4-vector is the space-time position 4-vector* (R^μ). Any set of 4 quantities that transform in the same way as the coordinates (ct, x, y, z) – *i.e.* according to the Lorentz transformation – forms a contravariant 4-vector.

The ***prototype covariant 4-vector is the space-time gradient of a 4-scalar field*** ϕ**:**

$$(\partial_0\phi\,,\partial_1\phi\,,\partial_2\phi\,,\partial_3\phi\,) = \left(\frac{1}{c}\frac{\partial\phi}{\partial t}\,,\frac{\partial\phi}{\partial x}\,,\frac{\partial\phi}{\partial y}\,,\frac{\partial\phi}{\partial z}\right)\,. \qquad (31)$$

The ***4-gradient operator*** **enjoys several notations:**

$$\frac{\partial}{\partial x^\mu} \stackrel{\text{def}}{=} \partial_\mu \stackrel{\text{def}}{=} (\quad)_{,\,\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\boldsymbol{\nabla}\right)\,. \qquad (32)$$

Any set of 4 quantities that transform in the same way as the coordinate derivatives, that is as $\partial_0, \partial_1, \partial_2, \partial_3$**, form a covariant 4-vector. The derivatives transform according to the inverse Lorentz transformation,** *i.e.* **from** $\mathcal{S}'$ **to** $\mathcal{S}$**.**

Reading: Jackson's Sect. 11.3B, ***4-Vectors*****, pp 526–7.**

09 Kinematics in Spacetime

Proper Time

The *proper time* is the time experienced by an observer and so measured along a worldline. It has the element $d\tau \equiv ds/c$; *i.e.* the space-time interval along the worldline, normalized to c to give the dimension of time.

Because the spatial distance element dr along a worldline is udt, with u the particle speed:

$$d\tau^2 \equiv ds^2/c^2 = dt^2 - dr^2/c^2$$

$$= dt^2 - (udt/c)^2 = dt^2\,\gamma^{-2}(u) \qquad (33)$$

$$i.e. \qquad dt/d\tau = \gamma(u) \qquad (34)$$

– a differential form of time-dilation relation, relating coordinate time t to proper time.

This is a much-used result, which keeps popping up in relativistic calculations. We'll generally use u to denote a particle speed and v for the speed of a reference frame.

Reading: Jackson, Sect. 11.3C, *Light Cone, Proper Time, and Time Dilation*, pp 527–9.

The Clock Hypothesis

The *clock hypothesis* states that the instantaneous rate of a clock depends on its instantaneous speed only, not on its acceleration. So $d\tau$ is the time interval shown by a clock.

This is a third hypothesis of Special Relativity, along with the *Principle of Relativity* (the laws of physics are the same in all inertial reference frames) and the existence of an *upper limiting speed* (usually – but not necessarily – identified with the vacuum speed of light) for the transfer of matter, energy, or information.

The 4-Velocity and 4-Acceleration

The *4-velocity* and *4-acceleration*, U & A, are 4-vectors along a worldline's tangent vector & principal normal (normal in the plane in which the tangent vector is turning):

$$U^\mu \equiv dx^\mu/d\tau = \gamma(u)(c\,,\mathbf{u})\,, \quad A^\mu \equiv dU^\mu/d\tau\,, \qquad (35)$$

with $\mathbf{u} \equiv d\mathbf{r}/dt$**, the usual 3-velocity of the particle.**

Division by the 4-scalar $d\tau$ **doesn't affect the 4-vector nature. Here** $d\tau \equiv ds/c$ **is the proper time between neighbouring events that are separated by spacetime interval** ds**. These definitions are not applicable for a null worldline (**$d\tau = 0$**).**

As $dt/d\tau = \gamma(u)$ **and** $(dx^\mu/dt) = (d/dt)(ct\,,\mathbf{r})$**, we get:**

$$(U^\mu) \equiv (dx^\mu/d\tau) = \gamma(u)(c\,,\mathbf{u})\,,$$
$$(U_\mu) = \gamma(u)(c\,,-\mathbf{u})\,. \qquad (36)$$

But the space-time breakup of (A^μ) is more complicated: With $\mathbf{a} \equiv d\mathbf{u}/dt$ denoting the 3-acceleration:

$$(A^\mu) \equiv \gamma(u)\frac{dU^\mu}{dt} = \gamma(u)\frac{d}{dt}\left(\gamma(u)c\,,\,\gamma(u)\mathbf{u}\right)$$

$$= \gamma^2(u)(0\,,\,\mathbf{a}) + (U^\mu)\frac{d\gamma(u)}{dt}$$

$$= \left(c\gamma(u)\frac{d\gamma(u)}{dt}\,,\;\; \gamma^2(u)\mathbf{a} + \mathbf{u}\gamma(u)\frac{d\gamma(u)}{dt}\right)\,. \qquad (37)$$

Note that:

$$\gamma(u)d\gamma(u)/dt = c^{-2}\gamma^4(u)u\,du/dt \qquad (38)$$

since $\gamma^{-2}(u) \equiv 1 - u^2/c^2$ gives $d\gamma(u) = \gamma^3(u)\,u\,du/c^2$.

At any instant when $u = 0$, both $\gamma(u) = 1$ and $d\gamma(u)/dt = 0$ hold. Then (A^μ) reduces to $(0\,,\,\mathbf{a}_0)$, with $\mathbf{a}_0$ the *proper 3-acceleration*:

***the acceleration in the instantaneous rest frame of the object and so the acceleration experienced by an observer on the object*.**

Properties of 4-velocity

(1) ***The 4-velocity is timelike*, because dx^μ connects events on a particle worldline, so $U_\mu U^\mu > 0$.**

(2) ***The 4-velocity is never zero*: when $\mathbf{u} = \mathbf{0}$, (U^μ) is $(c\,,\,\mathbf{0})$. There is no state of rest in space-time – you can't stop moving through time.**

(3) The squared magnitude of any 4-vector is a Lorentz scalar (invariant under LTs). In particular, from $(U^\mu) = \gamma(u)(c\,,\mathbf{u})$ and $(U_\mu) = \gamma(u)(c\,,\,-\mathbf{u})$, or from dividing $ds^2 = \eta_{\mu\nu}dx^\mu\,dx^\nu$ by $d\tau^2$:

$$U^2 = \eta_{\mu\nu}U^\mu U^\nu = U_\mu U^\mu = c^2\,. \tag{39}$$

That is, *all 4-velocities have magnitude c*.

(4) Because $U_\mu U^\mu = c^2$, we could say that everything is always hurtling at speed c through space-time*!* Ordinary objects move largely in time, not much in space; light moves entirely through space, with no experience of the passage of time. In a sense, time is the dominant dimension.

Properties of 4-acceleration

(1) ***The 4-acceleration and 4-velocity are orthogonal*** **at every point on a trajectory: taking** $d/d\tau$ **of** $U_\mu U^\mu = c^2$ **gives**

$$U_\mu A^\mu = 0 \,. \tag{40}$$

(2) As it's orthogonal to a timelike 4-vector, ***the 4-acceleration is spacelike*****.**

(3) ***The spatial part of*** (A^μ) ***is not generally parallel to*** **a: it also has a term parallel to u, except when** $udu/dt = 0$ **so that** $d\gamma(u)/dt = 0$**.**

(4) Because a scalar product is an invariant, and so can be evaluated in any inertial frame, we can use the instantaneous rest frame to get:

$$A_\mu A^\mu = -{a_0}^2 \,; \tag{41}$$

the sign characterizes (A_μ) ***as a spacelike 4-vector*****.**

(5) ***The 4-acceleration is zero if and only if*** **a = 0** **(so** $du/dt = 0$ **&** $d\gamma(u)/dt = 0$**).**

10 Dynamics in Spacetime

The 4-Momentum

The momentum and energy of a particle, of rest mass m_0 and 3-momentum p, combine into the *4-momentum* or *energy-momentum 4-vector*:

$$(P^\mu) = m_0\,(U^\mu) = (E/c\,,\,\mathbf{p})\,, \quad P_\mu = (E/c\,,\,-\mathbf{p})\,. \quad (42)$$

The squared magnitude of this is:

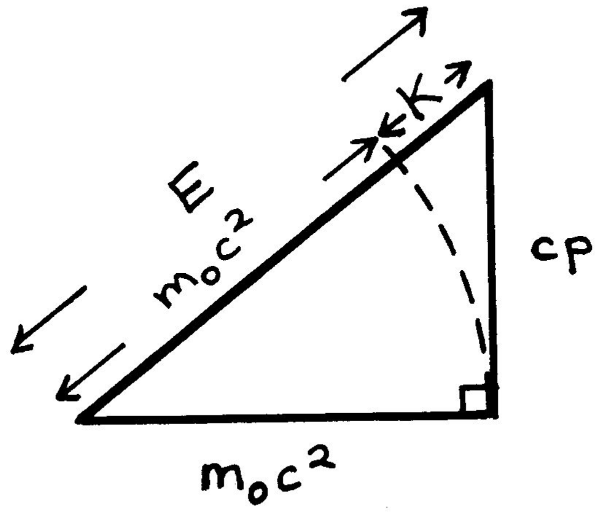
$$P_\mu P^\mu = \eta_{\mu\nu} P^\mu P^\nu = (E/c)^2 - p^2\,. \quad (43)$$

As this is a Lorentz invariant, we can evaluate it in the rest frame of the object to get $(E_0/c)^2$ or $(m_0 c)^2$, where $E_0 = m_0 c^2$ is the energy equivalent of the rest mass.

So follows the relativistic energy-momentum relation:

$$E^2 = (m_0 c^2)^2 + c^2 p^2\,, \quad (44)$$

replacing the classical $\mathrm{KE} = p^2/(2m_0)$, which holds when $p^2 << (m_0 c)^2$. In the ultra-relativistic limit of vanishing m_0, this quadratic relation becomes linear: $E = cp$.



The energy-momentum triangle.

The mass of a system measured in its centre of momentum frame (total 3-momentum zero) is the total system energy in this frame divided by c^2. It may not equal the sum of individual masses measured in other frames.

Reading: Jackson, Sect. 11.3B, *4-Vectors*, pp 526–7.

The 4-force

The *4-force* is defined as the rate of change of the energy-momentum 4-vector with respect to proper time:

$$(F^\mu) \equiv \left(\frac{dP^\mu}{d\tau}\right) = \left(\frac{d(E/c)}{d\tau}, \frac{d\mathbf{p}}{d\tau}\right). \qquad (45)$$

In the objects rest frame, the time component of the 4-force is zero unless the rest mass of the object is changing by energy/mass being directly added to or removed from it. For changing rest mass, the time component of the 4-force is the rate of change of rest mass, times c.

The spatial components of the 4-force are not generally equal to the components of the 3-force. This is because the 3-force is defined by the rate of change of 3-momentum with respect to coordinate time, $d\mathrm{p}/dt$, and the 4-force is defined by the rate of change of 4-momentum with respect to proper time, $(dF^\mu/d\tau)$.

For Newton's third law of motion to be applicable, both action and reaction 3-forces must be defined as the rate of change of 3-momentum with respect to the same time coordinate. Also, both must act at the same spatial point; *i.e.* **the law applies for bodies in contact only.**

11 Transformation of Electromagnetic Fields

Our notes for this topic are Jackson's Sect. 11.10, *Transformation of Electromagnetic Fields*, pp 558–61.

Note in particular:

- **The "pancaking" of the E field of a fast-moving charge.**
- **The *dilation* by a factor γ of the transverse E field of a fast-moving charge.**
- **The *contraction* by a factor $1/\gamma$ of the time interval during which the transverse E field of a fast-moving charge is appreciable.**
- **Jackson's diagrams on p 561.**

But for more details, see also: *Introductory Notes on Relativistic Electrodynamics.*

12 Charges and Currents in Spacetime

The 4-Current Density

Consider an electric charge density ρ^c and an electric 3-current density $\mathbf{j} = (j_x, j_y, j_z)$. We note that, because charge is invariant, charge density transforms between inertial frames inversely as the Lorentz contraction of volume:

$$\rho^c = \gamma(u)\rho^c{}_0 \,, \tag{46}$$

with $\rho^c{}_0$ the proper (rest-frame) charge density.

Suppose the current is a *convection current* from transport of ρ^c at 3-velocity u, so $\mathbf{j} = \rho^c\mathbf{u}$.

The source quantities combine into the *4-current density*, or *charge-current 4-vector*:

$$\begin{aligned}(J^\mu) &= (\rho^c c\,,\, \mathbf{j}) = \rho^c(c\,,\, \mathbf{u}) \\ &= \rho^c{}_0\gamma(u)(c\,,\, \mathbf{u}) = \rho^c{}_0(U^\mu)\,.\end{aligned} \tag{47}$$

As (U^μ) is a 4-vector and $\rho^c{}_0$ is invariant (always the rest-frame value), the last form shows that (J^μ) is a 4-vector.

The Covariant Continuity Equation

The usual *continuity equation*

$$(\partial \rho^c / \partial t) + \nabla \cdot \mathbf{j} = 0, \tag{48}$$

expressing charge conservation, becomes:

$$J^\alpha{}_{,\,\alpha} \stackrel{\text{def}}{=} \partial_\alpha J^\alpha = 0\,, \tag{49}$$

a vanishing divergence in 4 dimensions.

13 Gauge Invariance & the 4-Potential

Gauge transformations **are** ***transformations of the potentials that do not alter the fields*****; the fields are the physical quantities, defined directly in terms of forces, whereas potentials are tools for calculation (at least at the non-quantum level). All physical observables must be** ***gauge invariant*** **– unchanging under gauge transformations.**

In electrostatics, many problems are simplified by working with the scalar potential ϕ rather than the vector $\mathbf{E} = -\text{grad}\phi \equiv \nabla\phi$. The existence of ϕ is a direct consequence of the electrostatic equation $\nabla \times \mathbf{E} = 0$. Potentials differing by a constant, $\phi \to \phi + C$, correspond to the same E. This non-uniqueness of the electrostatic potential is the prototypical gauge invariance.

Similarly the condition $\nabla \cdot \mathbf{B} = 0$ enables the introduction of a vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$. And, because curl grad $\equiv 0$, a gauge transformation of adding the gradient of an arbitrary scalar function to A does not affect B.

Extending from electrostatics to electrodynamics, the two source-less Maxwell equations yield the forms (SI units):

$$\mathbf{B} = \nabla \times \mathbf{A}\,, \qquad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\,. \tag{50}$$

Substituting the former into Faraday's law shows that the curl of $(\mathbf{E} + \partial\mathbf{A}/\partial t)$ **vanishes, implying the existence of the spatial (3D) scalar** $\phi(\mathbf{r}\,,t)$**.**

The general gauge transformation now becomes not just $\phi \to \phi + C$ **but**

$$\mathbf{A} \to \mathbf{A} + \nabla\chi\,, \qquad \phi \to \phi - \frac{\partial \chi}{\partial t}\,, \tag{51}$$

where $\chi(\mathbf{r}\,,t)$ **is any function of position and time. The fields remain the same under the gauge transformation, and therefore Maxwell's equations are still satisfied. That is, Maxwell's equations have *gauge symmetry*.**

The 4-Potential

The *4-potential*, a 4-vector combining the electromagnetic scalar and vector potentials, can be introduced as:

$$(\Phi^\mu) = (\phi/c\,,\, \mathbf{A})\ , \qquad (\Phi_\mu) = (\phi/c\,, -\mathbf{A})\ , \tag{52}$$

in which ϕ and $\mathbf{A}$ are the usual scalar and vector (*i.e.* 3-scalar and 3-vector) potentials.

In the following sections, we'll mostly use Gaussian cgs units.

14 The Electromagnetic Field Tensor

Utilizing the electromagnetic 4-potential, we can define the *electromagnetic field tensor*:

$$F^{\mu\nu} = \partial^\mu \Phi^\nu - \partial^\nu \Phi^\mu \quad \textbf{or} \tag{53}$$

$$F_{\alpha\beta} = \partial_\alpha \Phi_\beta - \partial_\beta \Phi_\alpha = \Phi_{\beta,\alpha} - \Phi_{\alpha,\beta} \tag{54}$$

where

$$(\partial_\mu) = \left(\frac{\partial}{\partial x^\mu} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right) . \tag{55}$$

By its construction out of 4-vectors, $(F^{\mu\nu})$ is clearly a rank-2, antisymmetric 4-tensor.

Direct evaluation of the components of $(F_{\mu\nu})$ shows that the electric and magnetic fields,

$$\mathbf{E} = (E_x, E_y, E_z) \quad \textbf{and} \quad \mathbf{B} = (B_x, B_y, B_z) , \tag{56}$$

have been combined into the field tensor, with its covariant components displayed in:

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{57}$$

The contravariant components, $F^{\mu\nu} = \eta^{\mu\rho}\,\eta^{\nu\sigma}\,F_{\rho\sigma}$**, are displayed in:**

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{58}$$

The contravariant and covariant arrays are interchanged by putting E → −E.

The Dual Field Tensor

We use $\varepsilon^{\alpha\beta\gamma\delta}$**, the 4th-rank** ***Levi-Civita alternating tensor*****, having components** $+1, -1, 0$ **according as the indices are an even, odd, or no permutation of** $0, 1, 2, 3$**.**

The ***dual electromagnetic field tensor*** **is defined by**

$$(\mathcal{F}^{\mu\nu}) \equiv \varepsilon^{\mu\nu\gamma\delta} F_{\gamma\delta} \tag{59}$$

$$= \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{60}$$

The elements of this dual tensor may be obtained from $F^{\alpha\beta}$ by putting $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$. This is a special case of the *duality transformation*.

Reading: Jackson's Sect. 11.9, *Invariance of Electric Charge; Covariance of Electrodynamics*, pp 553–8.

15 Electrodynamics in Spacetime

Maxwell's equations have always been relativistic. By using tensor notation, we shall write them in explicitly covariant form meaning that their invariance under Lorentz transformations becomes obvious. Following Jackson, we use Gaussian cgs units (mostly).

We take the vacuum (or microscopic) form of Maxwell equations, in which all charges and currents are included; so macroscopic descriptions of materials (which incorporate atomic and molecular scale charges and currents via quantities such as permittivity) are not involved.

The Maxwell Equations

Consider *Faraday's law* and the *solenoidal rule* for magnetism, which are both source-free:

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{61}$$

These combine to form the 64 equations:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0\,, \qquad (62)$$

just 4 of which are independent. By antisymmetry of the field tensor, the equations reduce to an identity ($0 = 0$) or are rendered redundant, except for those with $\lambda\,,\mu\,,\nu$ = one of $1, 2, 3$ or $2, 3, 0$ or $3, 0, 1$ or $0, 1, 2$.

This tensor equation can be re-expressed as:

$$\varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta\,,\gamma} = 0\,, \qquad (63)$$

with $\varepsilon^{\alpha\beta\gamma\delta}$ the 4th-rank Levi-Civita alternating tensor, having components $+1, -1, 0$ according as the indices are an even, odd, or no permutation of $0, 1, 2, 3$.

The *Ampère-Maxwell law* and *Gauss's law*:

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}, \quad \nabla \cdot \mathbf{E} = 4\pi\rho^c\,, \qquad (64)$$

combine to form:

$$\partial_\nu F^{\mu\nu} = (4\pi/c) J^\mu\,. \qquad (65)$$

In SI units, the factor $4\pi/c$ is replaced by μ_0.

Lorentz 4-Force

The Lorentz 3-force density exerted on matter by the electromagnetic field is

$$\mathbf{f} = \rho^c \mathbf{E} + \mathbf{j} \times \mathbf{B} \,. \tag{66}$$

This is the spatial part of the *Lorentz 4-force*:

$$f_\mu = F_{\mu\nu} J^\nu \,, \tag{67}$$

as is readily checked.

For a convection current, $\mathbf{j} = \rho^c \mathbf{u}$ and $J^\mu = {\rho^c}_0 U^\mu$, so

$$f_\mu = {\rho^c}_0 F_{\mu\nu} U^\nu \,. \tag{68}$$

So we see that

$$f_\mu U^\mu = 0 \,. \tag{69}$$

That is, *the Lorentz 4-force is always orthogonal to the 4-velocity*.

Relativistic Equation of Motion

Newton's second law of motion for a particle of charge q in an electromagnetic field $(F_{\alpha\beta})$ has the relativistic form:

$$\frac{dP_\alpha}{d\tau} = q\, F_{\alpha\beta}\, U^\beta \tag{70}$$

where (P_α) is the particle's 4-momentum, (U^β) is its 4-velocity, and τ is its proper time.

In terms of an inertial observer's time (coordinate time) instead of proper time, the equation is

$$\frac{dP_\alpha}{dt} = q\, F_{\alpha\beta}\, \frac{dx^\beta}{dt}\,. \tag{71}$$

In a continuous medium, the 3-force density combines with the power density to form a 4-vector. The spatial part is the force per unit volume acting on a spatial volume element of the medium. The time component is $1/c$ times the power per unit volume transferred to that element.

16 The Lorenz Gauge

The *Lorenz condition*

$$\partial_\mu \Phi^\mu \equiv \Phi^\mu{}_{,\mu} = 0\,, \tag{72}$$

with (Φ_μ) the 4-potential, is obviously Lorentz invariant – hence it is well adapted to relativistic situations and is widely used in relativistic electrodynamics.

It is a partial gauge fixing of the electromagnetic vector potential. Within the *Lorenz gauge*, one can still make a gauge transformation:

$$\Phi_\mu \to \Phi_\mu + \partial_\mu \chi(\mathbf{r}\,,t) \tag{73}$$

where $\chi(\mathbf{r}\,,t)$ is a function satisfying $\partial_\mu \partial^\mu \chi = 0$, the standard wave equation (the equation of a massless scalar field in relativistic quantum theory). So:

***The Lorenz condition has the advantages of Lorentz invariance plus substantial gauge freedom*.**

The Lorenz condition is named after Ludvig Lorenz. Until recently, it was called the 'Lorentz condition'; see:

Jackson, J. D., 2008, Examples of the zeroth theorem of the history of science, *American Journal of Physics*, v. 76, pp 704–19; see Sec. II.

The Lorenz condition is generally used in calculations of time-dependent electromagnetic fields via retarded potentials (see later). In ordinary vector notation, the condition is:

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0\,. \qquad (74)$$

In SI units the $1/c$ becomes $1/c^2$, or $\mathbf{A}$ is replaced by $c\mathbf{A}$.

Wave equations derived from Maxwell's equations are simplified by the Lorenz condition $\partial_\alpha \Phi^\alpha = 0$ to:

$$\Box^2 \Phi^\alpha = (4\pi/c) J^\alpha \tag{75}$$

(in cgs units), where (J^α) is the 4-current density and

$$\Box^2 \equiv \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{76}$$

is the *wave operator* or *d'Alembertian operator* or just *d'Alembertian*.

In terms of the scalar and vector potentials, the electromagnetic wave equation separates into:

$$\Box^2 \phi = 4\pi \rho^c\,, \tag{77}$$

$$\Box^2 \mathbf{a} = -(4\pi/c)\mathbf{j}\,. \tag{78}$$

History: from Wikipedia (accessed April 2011): http://en.wikipedia.org/wiki/Lorenz gauge

"When originally published, Lorenz's work was not received well by James Clerk Maxwell. Maxwell had eliminated the Coulomb electrostatic force from his derivation of the electromagnetic wave equation since he was working in what would nowadays be termed the Coulomb gauge. The Lorenz gauge hence contradicted Maxwell's original derivation of the EM wave equation by introducing a retardation effect to the Coulomb force and bringing it inside the EM wave equation alongside the time varying electric field. Lorenz's work was the first symmetrizing shortening of Maxwell's equations after Maxwell himself published his 1865 paper. In 1888, retarded potentials came into general use after Heinrich Rudolf Hertz's experiments on electromagnetic waves. In 1895, a further boost to the theory of retarded potentials came after J. J. Thomson's interpretation of data for electrons (after which investigation into electrical phenomena changed from time-dependent electric charge and electric current distributions over to moving point charges)[1]."

[1] McDonald, K. T. (1997), The relation between expressions for time-dependent electromagnetic fields given by Jefimenko and by Panofsky and Phillips, *American Journal of Physics* v. 65 (11), 1074–76, doi:10.1119/1.18723.

17 The Inhomogeneous Wave Equations

The Maxwell equations in vacuum form, with sources ρ^c and $\mathbf{j}$, become, when expressed in terms of the vector and scalar potentials (in cgs units):

$$\nabla^2\varphi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho^c\,; \tag{79}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla \cdot \mathbf{A}\right) = -\frac{4\pi}{c}\mathbf{j}\,; \tag{80}$$

with $\mathbf{E} = -\nabla\varphi - c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\ \mathbf{A}$.

We have a coupled pair (scalar + vector) of inhomogeneous wave equations, with a messy mixture of φ and $\mathbf{A}$.

But under the Lorenz gauge condition

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0\,, \tag{81}$$

they neatly decouple into standard wave equations:

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho^c\,, \tag{82}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}\,. \tag{83}$$

In SI units, the RHSs are $-\rho^c/\varepsilon_0$ and $-\mu_0\mathbf{j}$.

In covariant form, with 4-current density $(J^\mu) = (c\rho^c\,,\,\mathbf{j}\,)$ and Lorenz condition $\partial_\mu\Phi^\mu = 0$:

$$\Box^2\Phi^\mu \overset{\text{def}}{=} \partial_\beta\partial^\beta\Phi^\mu \overset{\text{def}}{=} \Phi^{\mu,\beta}{}_\beta \tag{84}$$

$$= \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi^\mu \tag{85}$$

$$= -(4\pi/c)J^\mu\,(\text{cgs}) \quad \text{or} \quad -\mu_0 J^\mu \quad (\text{SI})\,, \tag{86}$$

with 4-potential $(\Phi^\mu) = (\phi\,,\mathbf{A})$ in cgs units and $(\phi\,,c\mathbf{A})$ in SI units.

Let R denote $|$r - r$'|$, the separation of the field point r from a typical source point r$'$. If there are no boundaries surrounding the sources, the solutions of the inhomogeneous wave equations are (in cgs units):

$$\phi(\mathbf{r},t) = \int \frac{\rho^c(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' + \frac{|\mathbf{r} - \mathbf{r}'|}{c} - t\right) d^3\mathbf{r}' dt' \qquad (87)$$

and

$$\mathbf{A}(\mathbf{r},t) = \int \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r} - \mathbf{r}'|}\delta\left(t' + \frac{|\mathbf{r} - \mathbf{r}'|}{c} - t\right) d^3\mathbf{r}' dt' \,. \qquad (88)$$

The delta functions ensure that sources contribute at retarded times only — that is, only where the spacetime source distribution intersects the past light cone of the observer.

Ref: Jackson's Sect. 6.3, pp 240–2, *Gauge Transformations, Lorentz gauge, Coulomb Gauge*.

***Our notes for this section include: Jackson's Sect. 6.4, pp 243–6,* Green Functions for the Wave Equation.**

For SI units:

$$\rho^c \to \frac{\rho^c}{4\pi\varepsilon_0}\,, \quad \mathbf{j} \to \frac{\mu_0}{4\pi}\,\mathbf{j}\,. \tag{89}$$

For Lorentz-Heaviside units:

$$\rho^c \to \frac{\rho^c}{4\pi}\,, \quad \mathbf{j} \to \frac{1}{4\pi}\,\mathbf{j}\,. \tag{90}$$

These retarded Lorenz-gauge potentials represent a superposition of spherical waves (of potentials and fields) traveling outward from the source, from the present into the future.

We shall see that mathematics in the Coulomb gauge is radically different — yet the physical outcome cannot differ.

18 The Coulomb / Radiation / Transverse Gauge

The Coulomb gauge (or radiation gauge or transverse gauge) is defined by the gauge condition:

$$\nabla \cdot \mathbf{A}(\mathbf{r}\,, t) = 0\,. \tag{91}$$

The inhomogeneous wave equations above for the potentials become (cgs units):

$$\nabla^2 \varphi = -4\pi \rho^c\,; \tag{92}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\frac{1}{c}\frac{\partial \varphi}{\partial t}\right) = -\frac{4\pi}{c}\,\mathbf{j}\,. \tag{93}$$

Note the great simplification of the first of the equations: it is decoupled from A and is no longer a wave equation but just the Poisson equation for the (generally non-static) scalar potential, with solution requiring only a spatial integration:

$$\phi(\mathbf{r}, t) = \int \frac{\rho^c(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \tag{94}$$

—note: no dash on the t in the integrand *!*

So the Coulomb-gauge scalar potential doesn't propagate — or shall we say that propagates with infinite speed throughout all space?

In the Coulomb gauge, the scalar potential at any time is the *instantaneous* Coulomb potential of the charge distribution *at that time.*

Hence the name "Coulomb gauge". In regions far from electric charge, φ approaches zero.

Once φ is known from solving Poisson's equation, the term involving φ in the wave equation for A becomes a source term. That term, *viz* $-c^{-1}\nabla(\partial\varphi/\partial t)$, is *irrotational* and can be cancelled by an irrotational part of j, as follows.

We write the 3-current density according to the Helmholtz decomposition (see below):

$$\mathbf{j} = \mathbf{j}_\ell + \mathbf{j}_t\,, \tag{95}$$

the sum of irrotational & solenoidal, or 'longitudinal' & 'transverse', current densities, with

$$\nabla\times\mathbf{j}_\ell = \mathbf{0} \quad \text{and} \quad \nabla\cdot\mathbf{j}_t = 0\,. \tag{96}$$

We choose

$$(4\pi/c)\mathbf{j}_\ell = c^{-1}\nabla(\partial\varphi/\partial t) \tag{97}$$

so the vector wave equation decouples from φ**, reducing to the standard wave equation for A,**

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}_t \tag{98}$$

but with source $\mathbf{j}_t$**.**

Notes on the Coulomb Gauge

(1) As the potentials (at least in classical, non-quantum, physics) are only calculational tools, with the field vectors and Lorentz force independent of the gauge, instantaneous propagation of the potentials does not violate casuality. As a check, one could calculate the fields from the Coulomb-gauge potentials.

(2) The Coulomb gauge is not Lorentz covariant. For potentials calculated in the Coulomb gauge, transformation to a new inertial frame will require a further gauge transformation if the Coulomb gauge condition is to be re-established.

(3) How much arbitrariness of gauge, or gauge freedom, remains within the Coulomb gauge? Adding a gauge function satisfying $\nabla^2\chi = 0$ has no effect, but the only solution of this equation that vanishes at infinity is $\chi(\mathbf{r}\,,\,t) = 0$. So no gauge freedom remains. The Coulomb gauge is said to be a complete gauge, unlike gauges where some gauge freedom remains, such as the Lorenz gauge.

(4) The Coulomb gauge is widely used in quantum chemistry and quantum-mechanical calculations in condensed matter physics, as well as in quantum electrodynamics.

Ref: Jackson's Sect. 6.3, pp 240–2, *Gauge Transformations, Lorentz gauge, Coulomb Gauge*.

***Exercise*: Jackson's Problem 6.20, pp 291–2: "An example of the preservation of causality and finite speed of propagation in spite of the use of the Coulomb gauge ...".**

The Helmholtz Decomposition

This topic is variously known as the Helmholtz decomposition, Helmholtz's theorem, and even as the fundamental theorem of vector calculus. It states that, in 3 dimensions, any vector field (that is sufficiently smooth and sufficently rapidly declining with distance) can be expressed as the sum of solenoidal (divergence-free) and irrotational (curl-free) parts.

This means that a vector field can be generated by a combination of a scalar and a vector potential, just as we have seen many times for the electric field. A little more formally:

Let F(r) be a vector field that is twice continuously differentiable in a volume V bounded by a surface S. It can be decomposed into curl-free and divergence-free parts:

$$\mathbf{F} = -\nabla\varphi + \nabla \times \mathbf{A} \tag{99}$$

with:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' - \frac{1}{4\pi}\int_S \frac{\mathbf{F}(\mathbf{r}')\cdot\mathrm{d}\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|} \qquad (100)$$

and

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' + \frac{1}{4\pi}\int_S \frac{\mathbf{F}(\mathbf{r}')\times\mathrm{d}\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|}\,. \qquad (101)$$

If V is all 3-space (*i.e.* unbounded), and F(r) vanishes sufficiently rapidly at infinity, then the surface integrals vanish.

In physics, the curl-free and divergence-free parts are often called 'longitudinal' & 'transverse' respectively. This comes from taking the 3D Fourier transform, $\tilde{\mathbf{F}}(\mathbf{k})$, of a vector field F(r) and decomposing it, at each point in k space, into two parts, 'longitudinal' & 'transverse' w.r.t. k, *i.e.* parallel & perpendicular to k:

$$\tilde{\mathbf{F}}(\mathbf{k}) = \tilde{\mathbf{F}}_\ell(\mathbf{k}) + \tilde{\mathbf{F}}_t(\mathbf{k}) \qquad (102)$$

with

$$\mathbf{k}\times\tilde{\mathbf{F}}_\ell(\mathbf{k}) = \mathbf{0} \quad \& \quad \mathbf{k}\cdot\tilde{\mathbf{F}}_t(\mathbf{k}) = 0\,. \qquad (103)$$

Taking the inverse Fourier transforms of these equations yields the Helmholtz decomposition:

$$\mathbf{F} = \mathbf{F}_\ell + \mathbf{F}_t \,, \tag{104}$$

$$\nabla \times \mathbf{F}_\ell = \mathbf{0} \quad \& \quad \nabla \cdot \mathbf{F}_t = 0 \,. \tag{105}$$

19 Relativistic Collisions

(i) Consider a collision between particles 1 & 2 of rest masses m and M. Let $\mathbf{P}_{(1,in)}$ & $\mathbf{P}_{(2,in)}$ denote their respective 4-momenta *before* the collision, becoming $\mathbf{P}_{(1,out)}$ & $\mathbf{P}_{(2,out)}$ *after* the collision. Conservation of momentum and total energy (rest + kinetic) are expressed by

$$\mathbf{P}_{(1,in)} + \mathbf{P}_{(2,in)} = \mathbf{P}_{(1,out)} + \mathbf{P}_{(2,out)} \,. \tag{106}$$

Let us use the initial rest frame of M, and take m to be the incident particle with velocity u and 3-momentum $\mathbf{p} = \gamma(\beta)m\mathbf{u}$ with $\beta \equiv u/c$; so:

$$\mathbf{P}_{(1,in)} \bullet \mathbf{P}_{(2,in)} = \gamma(\beta)mc(1, \mathbf{u}/c) \bullet Mc(1, \mathbf{0}) \tag{107}$$

$$= \gamma(\beta)mMc^2 \,. \tag{108}$$

where $\bullet$ denotes the 4D inner or dot product.

Rearranging this equation shows that the relative speed of each particle in the other's rest frame is given by:

$$\gamma(\beta) = \frac{\mathbf{P}_{(1,in)} \bullet \mathbf{P}_{(2,in)}}{mMc^2} \tag{109}$$

or

$$\beta^2 = 1 - \frac{(mMc^2)^2}{\left(\mathbf{P}_{(1,in)} \bullet \mathbf{P}_{(2,in)}\right)^2} \tag{110}$$

– see Jackson, Section 13.1, p 625 (3rd ed), just after Eq (13.2).

(ii) Suppose that the collision is *elastic*, meaning that the *rest mass* of each particle is unchanged by the collision – *i.e.* **no change in heat content if they are macroscopic particles and no change in energy levels if they are quantum particles.**

In elastic collisions, the 4D squared magnitudes of the 4-momenta are each unchanged in the collision:

$$\mathbf{P}_{(1,in)}{}^2 = m^2c^2 = \mathbf{P}_{(1,out)}{}^2 \,, \tag{111}$$

$$\mathbf{P}_{(2,in)}{}^2 = M^2c^2 = \mathbf{P}_{(2,out)}{}^2 \quad \textbf{(elastic case).} \tag{112}$$

Hence, taking the squared magnitude of both sides of the 4-momentum conservation equation at the start of this section shows that

$$\mathbf{P}_{(1,in)} \bullet \mathbf{P}_{(2,in)} = \mathbf{P}_{(1,out)} \bullet \mathbf{P}_{(2,out)} \quad \textbf{(elastic case).} \qquad (113)$$

This result has been called the *elastic collision lemma* – see W. Rindler, *Introduction to Special Relativity*, **Clarendon Press, Oxford, 1st ed. 1982, Section 28,** *Some four-momentum identities.*

If particles 1 & 2, of rest masses m & M, have relative speed v, then we always (not just in the elastic case) have

$$\mathbf{P}_1 \bullet \mathbf{P}_2 = \gamma(v) m M c^2 \,; \qquad (114)$$

this is readily established by using the rest frame of either particle.

Returning to the elastic case, we see that the elastic collision lemma is equivalent to the statement that the relative speed of two particles is unchanged in an elastic collision, a familiar result at the Newtonian level.

(iii) The 4-momentum transfer squared – a Lorentz invariant – is given by:

$$Q^2 = -\left[\mathbf{P}_{(1,in)} - \mathbf{P}_{(1,out)}\right]^2 \tag{115}$$

$$\equiv -\left[\mathbf{P}_{(1,in)} - \mathbf{P}_{(1,out)}\right] \bullet \left[\mathbf{P}_{(1,in)} - \mathbf{P}_{(1,out)}\right] \tag{116}$$

$$= \left(\mathbf{p}_{(1,in)} - \mathbf{p}_{(1,out)}\right)^2 - \left(E_{(1,in)} - E_{(1,out)}\right)^2 c^{-2}\,. \tag{117}$$

For elastic collisions, this evaluates, as quoted by Jackson in Section 13.1, p 625 (3rd ed), just before his Eq (13.2), to:

$$Q^2 = 4p^2 \sin^2(\theta/2) \qquad \textbf{(elastic collisions)}\,, \tag{118}$$

in terms of the momentum magnitude $p = \gamma(\beta)mc\beta$ of m in the rest frame of M; θ is the *scattering angle*, the angle through which the incident particle is deflected.

(iv) Consider elastic Coulomb (or Rutherford) scattering of particles 1 & 2 of rest masses m & M and charges $Z_1 e$ and $Z_2 e$ respectively. If we take M to be fixed in space, then m will follow a hyperbolic path under the inverse square law.

The inward and outward asymptotes to the path are separated by the scattering angle, θ. In the non-relativistic case, θ is found to be related to the impact parameter b and kinetic energy T of the incident particle m by (see *e.g.* Eisberg and Resnick, Appendix E & Section 4-2):

$$\tan(\theta/2) = \frac{Z_1 Z_2 e^2}{2Tb} \quad \textbf{(cgs units)}\,. \qquad (119)$$

A relativistic calculation gives the result quoted by Jackson at the start of problem 13.1 (p 655):

$$\tan(\theta/2) = \frac{Z_1 Z_2 e^2}{pc\beta b}\,, \quad \textbf{(cgs units)} \qquad (120)$$

involving the speed $c\beta$ and momentum magnitude, $p = \gamma(\beta) mc\beta$, of m.

(v) See Jackson's Problem 11.23 (p. 575), for use of the centre of momentum frame in treating collisions, and Problem 11.26 (p 576–7) for work on elastic collisions. The latter gives three expressions for the energy transfer, ΔE, in elastic collisions, including $\Delta E = Q^2/(2m)$.

Reading: Jackson's Sect. 13.1 (pp 625–7), *Energy Transfer in a Coulomb Collision Between Heavy Incident Particle and Stationary Free Electron; Energy Loss in Hard Collisions*.