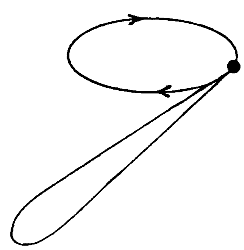
Notes on Maxwell's Equations and Relativity

R. R. Burman, May 2013

Contents (continued)



20 The Lorentz Force Density

The Lorentz force law, applicable to a single moving point charge, is here applied to a continuous distribution of charges and currents. Then Maxwell's equations are used to eliminate the charges and currents so as to re-express the force density in terms of the fields only.

From there, the idea is look for an analogue of the expression of the hydrostatic force density in a simple fluid as the negative pressure gradient, $\mathbf{f} = -\nabla p$, and its generalization

$$\mathbf{f} = -\nabla \cdot \overleftrightarrow{\mathbf{p}} \tag{121}$$

for a viscous fluid, with $\overleftrightarrow{\mathbf{p}}$ the pressure tensor of the fluid (negative of the stress tensor).

As this topic is generally encountered in a non-relativistic context, we'll use SI units. With an overdot denoting a partial time dervative:

(i) Faraday's law & the solenoidal condition:

$$\boldsymbol{\nabla} \times \mathbf{E} + \dot{\mathbf{B}} = \mathbf{0}\,, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0\,; \tag{122}$$

(ii) the Ampère-Maxwell & Gauss laws:

$$\boldsymbol{\mu_0}^{-1} \boldsymbol{\nabla} \times \mathbf{B} - \boldsymbol{\varepsilon}_0 \, \dot{\mathbf{E}} = \mathbf{j}\,, \quad \boldsymbol{\varepsilon}_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \boldsymbol{\rho}^c\,. \tag{123}$$

Manipulating the Lorentz Force Density

The Lorentz force $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ per unit volume acting on a volume distribution of charge and currents is:

$$\mathbf{f} = \rho^c \mathbf{E} + \mathbf{j} \times \mathbf{B} \,. \tag{124}$$

The Gauss and the Ampère-Maxwell laws enable elimination of the *sources* (ρ^c & $\mathbf{j}$) so that *only the fields* ($\mathbf{E}$ & $\mathbf{B}$) appear:

$$\mathbf{f} = \varepsilon_0 \left(\nabla \cdot \mathbf{E}\right) \mathbf{E} + {\mu_0}^{-1} \left(\nabla \times \mathbf{B}\right) \times \mathbf{B} - \varepsilon_0 \, \dot{\mathbf{E}} \times \mathbf{B} \,. \tag{125}$$

We can convert the time derivative term into a form that can, in part, be interpreted physically in terms of the Poynting vector:

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0 \,, \tag{126}$$

or $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ in cgs units. Differentiating and using Faraday's law to eliminate $\dot{\mathbf{B}}$ gives

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \dot{\mathbf{E}} \times \mathbf{B} - \mathbf{E} \times (\nabla \times \mathbf{E}) \,. \tag{127}$$

We now rewrite the Lorentz force density as:

$$\mathbf{f} = \varepsilon_0 (\nabla \cdot \mathbf{E})\,\mathbf{E} - \mu_0\varepsilon_0\,\dot{\mathbf{S}} + {\mu_0}^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B} - \varepsilon_0\mathbf{E} \times (\nabla \times \mathbf{E})\,. \tag{128}$$

On collecting terms with E and B we get:

$$\mathbf{f} = \varepsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})\right] + {\mu_0}^{-1} \left[(\nabla \cdot \mathbf{B})\mathbf{B} - \mathbf{B} \times (\nabla \times \mathbf{B})\right] - \mu_0\varepsilon_0\,\dot{\mathbf{S}}\,, \tag{129}$$

where an identically zero term in $\nabla \cdot \mathbf{B}$ has been added to create symmetry between E and B.

To rid ourselves of those nasty double cross product terms, we call on the so-often-useful 'curl identity':

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = \nabla(A^2/2) - (\mathbf{A} \cdot \nabla)\mathbf{A}\,. \tag{130}$$

We end up (so far as vector notation is concerned) with

$$\mathbf{f} = \varepsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}\right] + {\mu_0}^{-1} \left[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}\right] - \frac{1}{2}\nabla\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \mu_0\varepsilon_0\,\dot{\mathbf{S}} \tag{131}$$

– neat enough looking, but not strikingly simple after all that.

21 The Maxwell Stress Tensor

The Maxwell stress tensor is a 2nd-rank 3-tensor that represents the interaction between electromagnetic forces and mechanical momentum.

We start with the Lorentz force for a continuous distribution of charges and currents. The expression derived in the previous section contains all we need, but can be written more compactly by turning to tensor notation.

We introduce the Maxwell stress tensor $\overleftrightarrow{\mathbf{t}}$ with components:

$$t_{ij} \equiv \varepsilon_0 \left(E_i E_j - \delta_{ij} \frac{E^2}{2} \right) + \frac{1}{\mu_0} \left(B_i B_j - \delta_{ij} \frac{B^2}{2} \right) . \tag{132}$$

That is, the components are:

$$\begin{gathered} t_{11} = \frac{\varepsilon_0}{2} \left(2{E_1}^2 - E^2 \right) + \frac{1}{2\mu_0} \left(2{B_1}^2 - B^2 \right) , \\ \text{etc} , \\ t_{12} = \varepsilon_0 E_1 E_2 + {\mu_0}^{-1} B_1 B_2 , \\ \text{etc} . \end{gathered} \tag{133}$$

All terms but the time-derivative one in our vector-format expression for f coalesce into the divergence of this tensor:

$$\mathbf{f} + \mu_0\varepsilon_0\, \dot{\mathbf{S}} = \nabla \cdot \overleftrightarrow{\mathbf{t}}\,, \tag{134}$$

similar to the fluid dynamics equation of motion, except for the time derivative term.

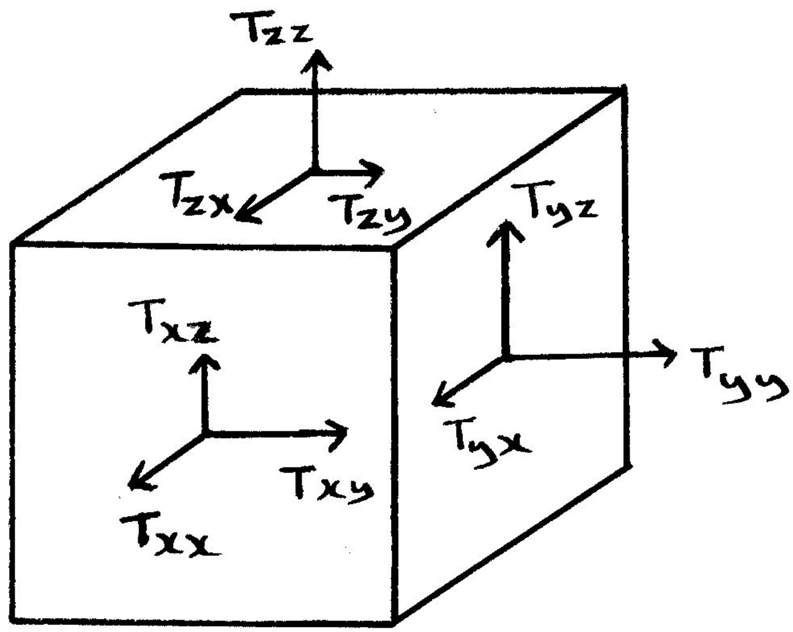
In Gaussian cgs units, commonly used in relativistic physics and in astronomy:

$$t_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}(E^2 + B^2)\delta_{ij}\right] \tag{135}$$

or, in dyadic notation with basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$:

$$\overleftrightarrow{\mathbf{t}} = \frac{1}{4\pi}\left[\mathbf{EE} + \mathbf{BB} - \frac{1}{2}(E^2 + B^2)\left(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}\right)\right]. \tag{136}$$

The Maxwell stress tensor has units of force per unit area (negative pressure): the ij element is the force per unit area parallel to the i axis exerted on a surface orthogonal to the j axis. The diagonal elements give the tension acting on area elements orthogonal to the corresponding axes.



These units are also units of momentum per unit area per second. The ij element is the flux of momentum parallel to the i axis crossing a surface normal to the j axis (in the negative direction) per unit time.

The divergence of the stress 3-tensor gives the dynamical effect on a fluid volume element of the rest of the system; we can imagine the external system removed and represented dynamically by this term.

Unlike the effect of pressure in an ideal gas, but like the effect of viscosity, an area element in the medium experiences an electromagnetic force in a direction that is not normal to the element. The shear is given by the off-diagonal elements of the stress tensor.

Reading: Jackson's Sect. 6.7, pp 258–62, *Poynting's Theorem and Conservation of Energy and Momentum for a System of Charged Particles and Electromagnetic Fields*.

The Magnetic Stress Tensor

If the field is magnetic only (as in magnetohydrodynamics with perfect conductivity, or, to a good approximation in motors), the stress tensor in SI units becomes:

$$(t_{ij}) = \frac{1}{\mu_0}\left(B_i B_j - \frac{B^2}{2}\delta_{ij}\right) . \tag{137}$$

For cylindrical objects, such as the rotor of a motor, this further simplifies to:

$$t_{r\theta} = \frac{1}{\mu_0}\left(B_r B_\theta - \frac{B^2}{2}\delta_{r\,\theta}\right) , \tag{138}$$

where r and θ denote the radial (outward from the cylinder) and tangential (around the cylinder) directions. It is the tangential force (shear) that spins the motor.

22 Magnetohydrodynamic Stresses

Magnetohydrodynamics, or MHD, is the study of the motion of a conducting fluid in a magnetic field. The fluid may be a liquid metal, such as the Earth's outer core, or a plasma. In the MHD approximation, a plasma is treated as a single fluid, with the constituent species recognized through the current density.

In MHD, the 'displacement current' is generally neglected: Ampère's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ is used, rather than the full Ampère-Maxwell relation. So electromagnetic waves are excluded and the subject is applicable to lower-frequency phenomena.

The force density on the fluid is

$$\mathbf{f} = \mathbf{j}\times\mathbf{B} = {\mu_0}^{-1}(\nabla\times\mathbf{B})\times\mathbf{B}\,, \tag{139}$$

on using the Ampère relation so as to express f entirely in terms of the field instead of sources and field.

To eliminate the double cross-product terms, we employ the 'curl identity':

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = \nabla(A^2/2) - (\mathbf{A} \cdot \nabla)\mathbf{A}\,, \qquad (140)$$

which converts the force density equation to:

$$\mu_0 \mathbf{f} = (\mathbf{B} \cdot \nabla)\,\mathbf{B} - \nabla(B^2/2)\,. \qquad (141)$$

This enables f to be expressed as the divergence of a cartesian tensor (Exercise: check this):

$$\mathbf{f} = \nabla\cdot \overleftrightarrow{\mathrm{t}} \quad \text{or} \quad f_k = t_{kn,n}\,, \qquad (142)$$

where $\overleftrightarrow{\mathrm{t}}$ is the magnetic stress tensor, with components

$$\mu_0 t_{mn} = B_m B_n - \delta_{mn}\,(B^2/2)\,. \qquad (143)$$

In dyadic form, with $\overleftrightarrow{\mathrm{I}}$ the unit tensor:

$$\mu_0 \overleftrightarrow{\mathrm{t}} = \mathbf{B}\,\mathbf{B} - (B^2/2)\,\overleftrightarrow{\mathrm{I}}\,. \qquad (144)$$

This linear combination of an outer product and the term in $\overleftrightarrow{\mathrm{I}}$ shows the cartesian tensor nature of $\overleftrightarrow{\mathrm{t}}$.

***Exercise*: study the behaviour of the components t_{mn} under 2D & 3D rotations of axes.**

When a stress tensor is in diagonal form, the elements are the principal stresses.

23 The Electromagnetic Stress-Energy Tensor

The 4D electromagnetic stress-energy tensor combines three key physical entities:

- **the Maxwell stress 3-tensor,**
- **the Poynting 3-vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ (cgs),**
- **the EM field energy density $(E^2 + B^2)/2$ (cgs).**

This symmetric 2nd-rank 4-tensor represents the flux density of the momentum 4-vector. In general relativity it contributes to the source term in Einstein's gravitational field equations. It is often called the electromagnetic energy-momentum tensor.

DE for the Stress-Energy Tensor

We again invoke the analogy with hydrostatics and hydrodynamics: the idea is to express the Lorentz force density, (f_μ), acting on a system of charges and currents as the divergence of a rank-2 tensor:

$$f^\mu = -T^{\mu\nu}{}_{,\nu} \,. \tag{145}$$

The Lorentz 4-force density is (as we have seen) related to the electromagnetic field tensor ($F_{\mu\nu}$) and the sources by the covariant equation:

$$f_\mu = F_{\mu\nu} J^\nu \,. \tag{146}$$

The above two equations show that the required tensor must satisfy the following differential equation, relating it to the electromagnetic field tensor and the current density 4-vector:

$$T^{\alpha\beta}{}_{,\beta} + F^{\alpha\beta} J_\beta = 0 \,. \tag{147}$$

This expresses conservation of linear momentum and energy in electromagnetic interactions.

The Stress-Energy Tensor

The tensor components shown may be checked by direct calculation of $T^{\mu\nu}{}_{,\nu}$:

$$(T^{\mu\nu}) = \frac{1}{4\pi} \begin{pmatrix} (E^2+B^2)/2 & S_x & S_y & S_z \\ S_x & -t_{xx} & -t_{xy} & -t_{xz} \\ S_y & -t_{yx} & -t_{yy} & -t_{yz} \\ S_z & -t_{zx} & -t_{zy} & -t_{zz} \end{pmatrix} , \tag{148}$$

where the t_{ij} are the components of the Maxwell stress tensor.

In SI units, J m^{-3}:

$$(T^{\mu\nu}) = \frac{1}{2}\begin{pmatrix} \varepsilon_0 E^2 + B^2/\mu_0 & S_x/c & S_y/c & S_z/c \\ S_x/c & -t_{xx} & -t_{xy} & -t_{xz} \\ S_y/c & -t_{yx} & -t_{yy} & -t_{yz} \\ S_z/c & -t_{zx} & -t_{zy} & -t_{zz} \end{pmatrix}. \tag{149}$$

The electromagnetic stress-energy tensor is related to the electromagnetic field tensor by the equation:

$$T^{\alpha\beta} = \frac{1}{4\pi}(F^{\alpha\sigma}\eta_{\sigma\tau}F^{\tau\beta} + \frac{1}{4}\eta^{\alpha\beta}F_{\sigma\tau}F^{\sigma\tau}) \tag{150}$$

where $(\eta_{\mu\nu})$ is the Minkowski metric tensor. This result can be checked by direct calculation (Exercise).

Ref: For a rather formal treatment:
Jackson's Sect. 12.10, pp 605–12, *Canonical and Symmetric Stress Tensors; Conservation Laws*. See particularly pp 608–10: *B. Symmetric Stress Tensors*.

24 Duality and Magnetic Monopoles

Our notes for this topic are Jackson's Sect. 6.11, *On the Question of Magnetic Monopoles*, pp 273–5.

Note in particular:

The two Maxwell vector equations with electric charge and current sources – the Ampère-Maxwell law and Gauss's law – are unaffected by the introduction of magnetic monopoles. In Gaussian cgs units:

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}\,, \quad \nabla \cdot \mathbf{E} = 4\pi\rho^{c}\,. \tag{151}$$

The two equations that are normally source-free – Faraday's law and the solenoidal rule for magnetism – acquire magnetic sources:

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \frac{4\pi}{c}\mathbf{j}^{m}\,, \quad \nabla \cdot \mathbf{B} = 4\pi\rho^{m}\,, \tag{152}$$

with $\mathbf{j}^m$ the magnetic current density and ρ^m the magnetic charge density.

The Lorentz 3-force exerted on a hypothetical particle with both electric charge q^e and magnetic charge q^m (a dyon) by an electromagnetic field is

$$\mathbf{f} = q^c(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B}) + q^m(\mathbf{B} - c^{-1}\mathbf{v} \times \mathbf{E})\,. \tag{153}$$

Magnetic monopoles and dyons are predicted by grand unified theories. Dyons were proposed as an alternative to quarks by:
J. Schwinger, *A magnetic model of matter*, **Science, 165(3895) 757–61, 1969.**

25 The Liénard Relativistic Radiation Formula

Our notes for this Section are:

Jackson's Sect. 14.2, *Total Power Radiated by an Accelerated Charge: Larmor's Formula and Its Relativistic Generalization*, pp 665–68.

Also see our introductory RED notes.

26 The Liènard-Weichert Potentials and Fields

Our notes for this Section are:

W. Rindler, *Introduction to Special Relativity*, Clarendon Press, Oxford, 1st ed. 1982; Sect. 40, *Potential and field of an arbitrarily moving charge*, pp 123–8.

See the REDreadings files.

27 Particle Drifts in Crossed E & B Fields

Consider a particle of charge e in a spatially and temporally uniform electromagnetic field. The equation of motion of an individual particle, with 3-velocity u in some inertial frame K, is

$$\frac{d\mathbf{p}}{dt} = e\,(\mathbf{E} + \mathbf{u} \times \mathbf{B}/c)\,. \qquad (154)$$

The component parallel to the magnetic field,

$$\frac{dp_{\parallel}}{dt} = e\,E_{\parallel}\,, \qquad (155)$$

describes uniform acceleration along B lines. A consequence is that plasmas near equilibrium generally have very small $E_{\parallel}$ outside of localized regions (which doesn't mean that $E_{\parallel}$ can be ignored). We'll consider the important special case of E $\perp$ B.

Transform to a frame K′ travelling at velocity $\mathbf{v}_E$ wrt the original frame. The equation of motion in the new frame is

$$\frac{d\mathbf{p}'}{dt'} = e\,(\mathbf{E}' + \mathbf{u}' \times \mathbf{B}'/c)\,. \qquad (156)$$

Suppose we choose $\mathbf{v}_E$ to be perpendicular to both E and B. Then the field transformation equations are

$$\mathbf{E}' = \gamma(v_E)\left(\mathbf{E} + c^{-1}\mathbf{v}_E \times \mathbf{B}\right) , \tag{157}$$

$$\mathbf{B}' = \gamma(v_E)\left(\mathbf{B} - c^{-1}\mathbf{v}_E \times \mathbf{E}\right) . \tag{158}$$

Suppose that the field magnitudes satisfy $E < B$, so we can choose

$$\mathbf{v}_E/c = (\mathbf{E} \times \mathbf{B})/B^2 , \tag{159}$$

$$v_E = E/B , \quad \gamma(v_E) = [1 - (E/B)^2]^{-1/2} . \tag{160}$$

Then the electric field in K′ is completely cancelled out and B′ is perpendicular to $\mathbf{v}_E$ with

$$\mathbf{B}_\perp{}' = \frac{1}{\gamma(v_E)}\mathbf{B} = \left(1 - \frac{E^2}{B^2}\right)^{1/2} \mathbf{B} . \tag{161}$$

Our notes for remainder of this Section are Jackson's Sect. 12.3, *Motion in Combined Uniform, Static Electric and Magnetic Fields*, pp 586–8.

28 Cherenkov Radiation

Our notes for this Section are:

J. Van Bladel, *Relativity and Engineering*, Springer- Verlag, Berlin, 1984:

Sect. 4.4, *Point Charge Moving Uniformly in a Dielectric Medium*, pp 113–6;

Sect. 4.5 *The Cerenkov Effect*, pp 116–20.

See the REDreadings files.

plus:

O. Heaviside, *Electromagnetic waves, the propagation of potential, and the electromagnetic effects of a moving charge*, published in four parts in *The Electrician, 1888–89*, reprinted in his *Electrical Papers, Vol. II*, Macmillan, New York and London, 1894, pp 490–9.

See the REDreadings files.

29 Radiation Reaction

Our notes for this topic are Jackson's:
Sect. 16.1, *Introductory Considerations*, pp 745–7;
Sect. 16.2, *Radiative Reaction Force from Conservation of Energy*, pp 747–50.

End of MER Notes